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Nonlinear Diffusive Phenomena of Solutions for the System of Compressible Adiabatic Flow through Porous Media

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1. INTRODUCTION

$$
\begin{aligned}
&v_t - u_x = 0\\
&u_t + p(v, s)_x = -\alpha u, \ \alpha > 0\\
&\left[e(v, s) + \frac{u^2}{2}\right]_t + (pu)_x = -\alpha u^2
\end{aligned}
\tag{1.0}
$$

which can be used to model the adiabatic gas flow through porous media. Where v denotes the specific volume, u is velocity, s stands for entropy, p denotes pressure with $p_v < 0$ for $v > 0$, e denotes the specific internal energy for which $e_s \neq 0$ and $e_v + p = 0$ holds due to the second law of thermodynamics. The system (1.0) is equivalent to the following system for smooth solutions

$$
\begin{aligned}
&v_t - u_x = 0\\
&u_t + p(v, s)_x = -\alpha u, \qquad \alpha > 0\\
&s_t = 0
\end{aligned}
\tag{1.1}
$$

which is strictly hyperbolic with eigenvalues $\lambda_1 = -\sqrt{-p_v}$, $\lambda_2 \equiv 0$, and $\lambda_3 = \sqrt{-p_v}$.

The aim of this paper is to investigate the role of damping mechanism, in particular, the influence to the asymptotic behavior of the processes in consideration.

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For the case of isentropic flow, namely, $s(x, t) \equiv$ constant, it has been proved in [HLI1] and [HLI2] that the solution of the Cauchy problem

$$\begin{gathered} v_t - u_x = 0 \\ u_t + p(v)_x = -\alpha u, \qquad \alpha > 0, \quad p'(v) < 0 \qquad \text{for} \quad v > 0 \\ v(x, 0) = v_0(x), \\ u(x, 0) = u_0(x) \qquad \text{with} \qquad \lim_{x \to \mp\infty} (v_0(x), u_0(x)) = (v^{\mp}, u^{\mp}) \end{gathered} \tag{1.2}$$

can be described by the solution of the following problem,

$$\begin{gathered} v_t = -\frac{1}{\alpha} p(v)_{xx} \\ u = -\frac{1}{\alpha} p(v)_x \\ v(x, 0) = \bar{v}^*(x + d_0), \end{gathered} \tag{1.3}$$

time-asymptotically, where $\bar{v}^*$ is the similar solution of $(1.3)_1$ with $\bar{v}^*(\pm\infty) = v^{\pm}$, $d_0 \in R^1$ is a constant. The system in (1.3) is obtained from (1.2) by approximating the momentum equation in $(1.2)_2$ with Darcy's law. Moreover, the L_2-norm and L_∞-norm of the difference between these two solutions tend to zero with a rate $t^{-1/2}$ as time tends to infinity ([HLI1]). This shows that certain nonlinear diffusive phenomena occur for the solutions of (1.2) which is caused by the damping mechanism.

For the system (1.1), the corresponding simplified system takes the form

$$\begin{gathered} v_t = -\frac{1}{\alpha} p(v, s)_{xx} \\ u = -\frac{1}{\alpha} p(v, s)_x \\ s_t = 0. \end{gathered} \tag{1.4}$$

We will compare the solution of (1.1) with initial data $(u(x, 0), v(x, 0), s(x, 0))$ to the solution of (1.4) with suitable initial data and prove that the L_∞-norm of the difference between these two solutions tends to zero as time tends to infinity. This shows that the large time behavior of solutions for the nonlinear hyperbolic system (1.1) can be well approximated by the corresponding simplified nonlinear diffusion equations and certain nonlinear diffusive phenomena can be expected to occur also, as in the isentropic case, caused by the damping mechanism.

Consider the initial data

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \qquad s(x, 0) = s_0(x). \tag{1.5}$$

It has been proved in [HS] that the solution of (1.1), (1.5) can be described by the solution of (1.4) with initial data $(v(x, 0), s(x, 0))$ time-asymptotically, provided

$$\lim_{x \to \mp\infty} u_0(x) = 0, \qquad \lim_{x \to \mp\infty} v_0(x) = \bar{v} > 0, \qquad \lim_{x \to \mp\infty} s_0(x) = \bar{s}, \tag{1.6}$$

where $\bar{v}$ and $\bar{s}$ are constants.

We deal with more general initial data (1.5) in the present paper, namely,

$$\lim_{x \to \mp\infty} u_0(x) = u^{\mp}, \qquad \lim_{x \to \mp\infty} v_0(x) = v^{\mp} > 0, \qquad \lim_{x \to \mp\infty} s_0(x) = \bar{s}, \tag{1.7}$$

where v^- is not equal to v^+. This causes essential difficulties which will be explained later. For definiteness, we assume $v^- < v^+$. The case $v^- > v^+$ can be discussed similarly. Furthermore, it holds that

$$\begin{aligned} v_0(x) - v^+ &= O(x^{-k_1}), && \text{as} \quad x \to +\infty \\ v_0(x) - v^- &= O((-x)^{-k_2}), && \text{as} \quad x \to -\infty \end{aligned} \tag{1.8}$$

for certain positive constants $k_i > 1 (i = 1, 2)$, while $(s_0(x) - \bar{s})$ possesses a compact support, say

$$\text{support}(s_0(x) - \bar{s}) \subset [-N, N] \tag{1.9}$$

for a positive constant N. For simplicity, we consider a typical form of $p(v, s)$ for gas dynamics from now on, namly, $p(v, s) = (\gamma - 1)\, v^{-\gamma} e^s$, $(1 < \gamma < 3)$. The following results can be established for general $p(v, s)$ which will be clarified later.

It is known that there exists a similarity solution $v^*(\eta)$, $\eta = (x/\sqrt{t+1})$ for the equation $v_t = -(1/\alpha)\, p(v, \bar{s})_{xx}$, satisfying the boundary condition $v^*(\eta) \to v^{\mp}$ as $\eta \to \mp\infty$, where $\bar{s}$ is the constant in (1.7).

Due to (1.8), (1.9), $v^- < v^+$ and the facts (see [DP2], [AP1], [AP2]) that $v^*(\eta) - v^- = O(\text{erfc}(-\eta/2\sqrt{-p_v(v^-, \bar{s})}))$ as $\eta \to -\infty$ and $v^*(\eta) - v^+ = O(\text{erfc}(\eta/2\sqrt{-p_v(v^+, \bar{s})}))$ as $\eta \to +\infty$, and $v^*_\eta(\eta) > 0$ (in the case of $v^- < v^+$) for $\eta \in R$, we can choose $x^* \in R$ uniquely such that

$$\int_{-\infty}^{+\infty} [v_0(x) - e^{(1/\gamma)(s_0(x) - \bar{s})} v^*(x + x^*)]\, dx = \frac{u^+ - u^-}{-\alpha}. \tag{1.10}$$

It is clear that

$$p(e^{(1/\gamma)(s_0(x)-\bar{s})}v^*(x+x*),\ s_0(x))=p(v^*(x+x^*),\ \bar{s}).$$

Denote the solution of (1.1), (1.5) by $(u_1(x,t), v_1(x,t), s_1(x,t))$, and the solution of (1.4) with $v(x,0)=e^{(1/\gamma)(s(x)-\bar{s})}v^*(x+x^*)$, $s(x,0)=s_0(x)$ by $(u_2(x,t), v_2(x,t), s_2(x,t))$. It is obvious that

$$s_1(x,t)\equiv s_2(x,t)\equiv s(x)\equiv s_0(x).$$

Take any smooth function $m_0(x)$ with a compact support and

$$\int_{-\infty}^{+\infty} m_0(x)\,dx=1,$$

and define

$$m(x,t)\equiv -\frac{u^+-u^-}{\alpha}\,m_0(x)\,e^{-\alpha t}.$$

Let

$$\hat{u}(x,t)\equiv u^-e^{-\alpha t}+\int_{-\infty}^{x} m_t(\xi,t)\,d\xi$$

$$w(x,t)\equiv v_1(x,t)-v_2(x,t)-m(x,t)$$

$$z(x,t)\equiv u_1(x,t)-u_2(x,t)-\hat{u}(x,t).$$

It follows by (1.1) and (1.4) that

$$w_t-z_x=0$$

$$z_t+[p(w+v_2+m,s)-p(v_2,s)]_x+\alpha z-\frac{1}{\alpha}\,p(v_2,s)_{xt}=0,$$

where $s=s(x)\equiv s_0(x)$ is known.

We will seek the solution w and z satisfying $w(\pm\infty,t)=z(\pm\infty,t)=0$. Introducing $y(x,t)=\int_{-\infty}^{x} w(\xi,t)\,d\xi$, the above system can be reduced into a single equation for y since $y_x=w$, $y_t=z$, due to (1.1) and (1.4). Namely

$$y_{tt}+[p(y_x+v_2+m,s)-p(v_2,s)]_x+\alpha y_t-\frac{1}{\alpha}\,p(v_2,\ s)_{xt}=0. \quad (1.11)$$

where v_2 satisfies $(1.4)_1$ and $v_2(x,0)=e^{(1/\gamma)(s(x)-\bar{s})}v^*(x+x^*)$, $s=s(x)=s_0(x)$.

Ignore the subscript with v and assume that $\alpha=1$ for convenience, we study the following initial-value problem

$$y_{tt}+[p(y_x+v+m, s)-p(v, s)]_x+y_t-p(v, s)_{xt}=0 \tag{1.12}$$

$$y(x, 0)=y_0(x),\ y_t(x, 0)=y_1(x), \tag{1.13}$$

where

$$y_0(x)=\int_{-\infty}^{x}[v_0(x)-e^{(1/\gamma)(s(x)-\bar{s})}v^*(x+x^*)-m(x, 0)]\,dx$$

$$y_1(x)=u_0(x)+[p(e^{(1/\gamma)(s(x)-\bar{s})}v^*(x+x^*), s(x))]_x-\hat{u}(x, 0)$$

while $v(x, t)$ satisfies

$$\begin{aligned} v_t&=-p(v, s)_{xx}\\ v(x, 0)&=e^{(1/\gamma)(s(x)-\bar{s})}v^*(x+x^*) \end{aligned} \tag{1.14}$$

with $s=s(x)=s_0(x)$.

It is clear that $y(\pm\infty, t)=0$ in view of (1.10), $w_t=z_x$ and the definition of $m(x, t)$. In addition, $y_1(\pm\infty)=0$ due to the fact that $v_x^*\to 0$ as $x\to\pm\infty$ (see [DP2, AP1, AP2]).

For any given initial data $(v_0(x), u_0(x))$ such that $y_0(x)\in H^3(R)$, $y_1(x)\in H^2(R)$, we prove that (1.12)–(1.13) has a unique smooth solution in the large in time in the space X_3 provided that the initial data are small (the precise description for the smallness will be given later and the definition of X_3 will be given in Section 2). Furthermore, the solution y and its derivatives y_t, y_x decay to zero in the L_∞-norm as $t\to+\infty$ which implies that the system (1.1) is accurately approximated by (1.4) time-asymptotically since the function $m(x, t)$ and $\hat{u}(x, t)$ decay to zero exponentially fast. For convenience, we only give the proof for the case when $u^-=u^+=0$ in which $m(x, t)\equiv 0$, $\hat{u}(x, t)\equiv 0$ and (1.10) becomes

$$\int_{-\infty}^{+\infty}[v_0(x)-e^{(1/\gamma)(s(x)-\bar{s})}v^*(x+x^*)]\,dx=0.$$

The general case can be treated in a similar way by using the properties of $m(x, t)$ and $\hat{u}(x, t)$.

In contrast with the case of $v^+=v^-$ discussed in [HS] or the case of $s_0(x)\equiv\bar{s}$ discussed in [HLI], certain essential difficulties occur when we deal with the case of $v^+\neq v^-$ and $s_0(x)\not\equiv\bar{s}$. For the case of $s_0(x)\equiv\bar{s}$, there exists a similarity solution $v^*(\eta)$, $\eta=x/\sqrt{t+1}$ of $(1.4)_1$, with $v^*(\mp\infty)=v^{\mp}$, for which the estimates about decay rate of L^2-norm on the corresponding required derivatives have been well established in [HLI1] and [HLI2]. Therefore, one may compare the solution v_1 of (1.1) with v^* directly, instead of the solution v_2 of $(1.4)_1$, with the help of those decay estimates. Unfortunately, there is no similarity solution for $(1.4)_1$ in the case of $s_0(x) \not\equiv \bar{s}$. However, the function $e^{(1/\gamma)(s(x)-\bar{s})}\bar{v}(s(x)$ satisfies (1.7)) is just a special solution of $(1.4)_1$ which is used to be compared with the solution of $(1.4)_1$ to obtain the result in [HS] for the case when $v^- = v^+ = \bar{v}$. In order to prove that the corresponding required derivatives of the solution $v(x, t)$ of (1.14) have the similar properties to those derivatives of v^*, we consider the following problem

$$\begin{aligned} v_t &= -p(v, s)_{xx} \\ v(x, 0) &= e^{(1/\gamma)(s(x)-\bar{s})} v^*(x + x_0), \end{aligned} \tag{1.15}$$

where $x_0 \in R^1$, which will be chosen later to make the solution of (1.15) satisfy certain expected properties, and $v^*(\eta)$, $\eta = x/\sqrt{t+1}$, is the unique similarity solution, with boundary condition $v^*(\mp\infty) = v^{\mp}$, for the equation

$$v_t = -p(v, \bar{s})_{xx}. \tag{1.16}$$

Denote the solution of (1.15) by $\tilde{v}(x, t)$, we would compare the solution $v(x, t)$ of (1.14) with $\tilde{v}(x, t)$. First, by comparing $\tilde{v}(x, t)$ with a function $e^{(1/\gamma)(s(x)-\bar{s})} v^*((x + x_0 + a_0)/\sqrt{t+1})$ (a_0 is a constant), we obtain the global existence and uniqueness for (1.15) and the L^2-estimates on the corresponding required derivatives of $\tilde{v}$. Then, we compare the solution $v(x, t)$ with $\tilde{v}(x, t)$ to get the expected decay estimates on the corresponding derivatives of $v(x, t)$. Section 2 is devoted to study these nonlinear diffusion equations.

For carrying on this approach, we need the following condition which makes some restrictions on $s_0(x)$ and $v^{\pm}$.

Condition V. In addition to (1.7)–(1.9), $s_0(x)$, v^+ and v^- are given so that there are constants $k_0 \geqslant N$, $a_0 \geqslant 0$ and $x_0 \in R$ such that it holds that

$$\tilde{v}(x, t) \leqslant v^*\left(\frac{x + x_0 + a_0}{\sqrt{t+1}}\right) \qquad \text{for} \quad |x| \geqslant k_0 \tag{1.17}$$

if $\tilde{v}(x, t)$ is a positive solution of (1.15).

Remark 1.1. The Condition V is based on the facts that $v^*((x + x_0)/\sqrt{t+1})$ satisfies $(1.15)_1$ and $(1.15)_2$ for $|x| \geqslant N$ since support $(s(x) - \bar{s}) \in [-N, N]$, and $v^*(\eta)$ is monotonically increasing due to $v^- < v^+$ (see [AP1, AP2, DP2]).

Remark 1.2. It is very easy to check the Condition V in the case of $s_0(x) \equiv \bar{s}$, since the solution $\tilde{v}(x, t)$ of (1.15) is just $v^*((x + x_0)/\sqrt{t+1})$.

Another trivial case appears when $v^+ = v^- = \bar{v}$, for which $v^*(x/\sqrt{t+1}) = \bar{v}$ and the function $e^{(1/\gamma)(s(x)-\bar{s})}\bar{v}$ is the solution of (1.15). Due to support $\{s(x)-\bar{s}\} \subset [-N, N]$, (1.17) holds obviously.

Remark 1.3. We may propose the condition V′ by changing (1.17) into

$$\tilde{v}(x, t) \geqslant v^* \left(\frac{x + x_0 - a_0}{\sqrt{t+1}} \right) \qquad \text{for} \quad |x| \geqslant k_0 \tag{1.18}$$

and get the same result.

When $s_0(x)$ satisfies certain conditions, it is possible to construct a function $e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*((x+x_0+a_0)/\sqrt{t+1})$ as an upper solution of (1.15) or a function $e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*((x+x_0-a_0)/\sqrt{t+1})$ as a lower solution of (1.15). For instance, in the case of $s(x) \leqslant \bar{s}$, $x \in [-N, N]$, it can be verified that the function $e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*((x-N+2N)/\sqrt{t+1})$ is an upper solution of (1.15) with $x_0 = -N$, the condition V is satisfied then. While the function $e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*((x-N)/\sqrt{t+1})$ is a lower solution of (1.15) with $x_0 = N$, namely, the condition V′ is satisfied also. The case when $s(x) \geqslant \bar{s}$, $x \in [-N, N]$ can be handled similarly. Another typical case is $s(x) \leqslant \bar{s}$ for $x \leqslant x'$ and $s(x) \geqslant \bar{s}$ for $x \geqslant x'$ (or $s(x) \geqslant \bar{s}$ for $x \leqslant x'$ and $s(x) \leqslant \bar{s}$ for $x \geqslant x'$). We assume $x' = 0$ for convenience. It is not difficult to verify that the function $e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*(x/\sqrt{t+1})$ is a lower (or upper) solution of (1.15) with $x_0 = 0$.

Remark 1.4. For more general form of $p(v, s)$ when $p(v, s) = f(v)\, g(s)$ with $f'(v) < 0$, $g(s) > 0 (v > 0, s \in R)$ and suitable smoothness on f and g, we can choose $f^{-1}[f(v^*(x+x^*))\, g(\bar{s})/g(s_0(x))]$ and $f^{-1}[f(v^*(x+x_0))\, g(\bar{s})/g(s_0(x))]$, instead of $e^{(1/\gamma)(s_0(x)-s)}v^*(x+x^*)$ in $(1.14)_2$ and $e^{(1/\gamma)(s_0(x)-\bar{s})} \cdot v^*(x+x_0)$ in $(1.15)_2$, respectively. The similar result as presented in Theorem 3.2 can be obtained then by the analysis used in the present paper.

2. Nonlinear Diffusion Equations

We study the following two problems in this section

$$\begin{aligned} v_t &= -p(v, s)_{xx} \\ v(x, 0) &= e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*(x+x^*), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} v_t &= -p(v, s)_{xx} \\ v(x, 0) &= e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*(x+x_0), \end{aligned} \tag{2.2}$$

where $x_0 \in R$ is the constant in condition V and $s = s(x) = s_0(x)$ as mentioned in Section 1, while $x^* \in R$ is determined uniquely by

$$\int_{-\infty}^{\infty} [v_0(x) - e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*(x+x^*)]\, dx = 0.$$

We will prove the global existence of smooth solutions for (2.1) and (2.2) respectively and establish the L^2-estimates on the corresponding derivatives of these solutions.

It is known (see [DP2], [AP1], [AP2]) that there exists a unique similarity solution $v^*(\eta)$, $\eta = x/\sqrt{t+1}$, satisfying $v^*(\mp\infty) = v^{\mp}$, for the equation (1.16), namely, $v^*(\eta)$ satisfies

$$\begin{aligned} [-p(v^*(\eta), \bar{s})]_{\eta\eta} + \tfrac{1}{2}\eta v^*_\eta &= 0 \\ v^*(\mp\infty) &= v^{\mp}. \end{aligned} \tag{2.3}$$

Moreover, it has been shown also that

$$\begin{aligned} v^*(\eta) - v^- &= O\left(\operatorname{erfc} \frac{-\eta}{2\sqrt{-p_v(v^-, \bar{s})}}\right) \qquad \text{as} \quad \eta \to -\infty \\ v^*(\eta) - v^+ &= O\left(\operatorname{erfc} \frac{\eta}{2\sqrt{-p_v(v^+, \bar{s})}}\right) \qquad \text{as} \quad \eta \to +\infty. \end{aligned} \tag{2.4}$$

$v^*_\eta(\eta) > 0$ for $\eta \in R$ if $v^- < v^+$ which implies that

$$v^- < v^*(\eta) < v^+ \tag{2.5}$$

and

$$v^*(\eta) \equiv \bar{v} \qquad \text{if} \quad v^- = v^+ = \bar{v}. \tag{2.6}$$

We look for a positive smooth solution $\tilde{v}(x, t)$ for (2.2) first.
Define

$$\hat{v}(x, t) = e^{(1/\gamma)(s(x)-\bar{s})} \cdot v^*\left(\frac{x + x_0 + a_0}{\sqrt{t+1}}\right), \tag{2.7}$$

where x_0 and a_0 are the constants in condition V, $v^*(\eta)$ is the solution of (2.3).

It is easy to verify, due to the form of $p(v, s)$, that

$$p(\hat{v}(x, t), s(x)) = p\left(v^*\left(\frac{x + x_0 + a_0}{\sqrt{t+1}}\right), \bar{s}\right). \tag{2.8}$$

Let $\tilde{v}(x, t) > 0$ be a smooth solution of (2.2) and $\theta(x, t) = \tilde{v}(x, t) - \hat{v}(x, t)$, it holds that

$$\theta_t = -p(\hat{v} + \theta, s)_{xx} - e^{(1/\gamma)(s(x) - \bar{s})} \cdot v_t^* \tag{2.9}$$

$$\theta(x, 0) = e^{(1/\gamma)(s(x) - \bar{s})} \cdot [v^*(x + x_0) - v^*(x + x_0 + a_0)], \tag{2.10}$$

where $\hat{v}$ is defined in (2.7).

We seek a smooth solution $\theta(x, t) \in C^2(t \geqslant 0, x \in R)$ and

$$\|\theta(\cdot, t)\|_2 \equiv |\theta(\cdot, t)|_{C^2} + |\theta_t(\cdot, t)|_{C^1} + |\theta_{tt}(\cdot, t)|_{C^0} < \infty \tag{2.11}$$

where

$$|f(\cdot)|_{C^l} \equiv \sum_{0 \leqslant j \leqslant l} \sup_R |d^j f(x)/dx^j|.$$

The uniqueness of such a solution $\theta(x, t)$ satisfying $\hat{v}(x, t) + \theta(x, t) > 0$($x \in R$, $t \in [0, T]$ for any $T > 0$) can be easily obtained by the Maximum principle. Therefore, $\tilde{v}(x, t) = \hat{v}(x, t) + \theta(x, t)$ would be the unique positive solution of (2.2) if one could show that there exists a solution $\theta(x, t)$ for (2.9) and (2.10) such that $\hat{v}(x, t) + \theta(x, t) > 0$. We show the existence of such a solution $\theta(x, t)$ next.

By using Sobolev's lemma, it is known that

$$|f(\cdot)|_{C^l} \leqslant K\|f(\cdot)\|_{H^{l+1}}. \tag{2.12}$$

Thus, using the L^2-energy method we will solve the Cauchy problem (2.9), (2.10) in the Banach space X_3 defined by

$$X_m = \left\{ \begin{matrix} \theta(t) \in L_\infty(t; H^m), \theta_t(t) \in L_\infty(t; H^{m-1}), \\ \theta_{tt}(t) \in L_\infty(t; H^{m-2}), 0 \leqslant t \leqslant T \text{ for any } T > 0 \end{matrix} \right\}. \tag{2.13}$$

Assume

$$(s_0(x) - \bar{s}) \in H^5(R). \tag{2.14}$$

It can be obtained that

$$\|\theta(\cdot, 0)\|_{H^5}^2 \leqslant K a_0^2 \{[v_\eta^*(0)]^2 + \|s_x\|_{H^4}^2\} \tag{2.15}$$

by a similar argument as used in [HLI1] and [HLI2] to get the L^2-estimates on the corresponding derivatives of the similarity solution. Where $K > 0$ is a constant depending only on the quantities b, $d_i (i = 1, 2, 3)$, $\bar{s}$ and γ, defined as follows

$$b = \sup_{x \in R} e^{(1/\gamma)(s(x) - \bar{s})},$$

$$d_1 = \inf_{v \in [v^-, v^+]} [-p_v(v, \bar{s})] > 0,$$

$$d_2 = \sup_{v \in [v^-, v^+]} [-p_v(v, \bar{s})] > 0,$$

$$d_3 = \sup_{v \in [v^-, v^+]} \sum_{i=1}^{5} \left| \frac{\partial^i p(v, \bar{s})}{\partial v^i} \right|.$$

It can be proved, by a routine manner, that the classical local existence theorem gives the solution for the Cauchy problem (2.9)–(2.10) in the space X_3 locally in time. For the global existence in $t > 0$ we only need the a priori estimates in the norm (2.11) for which the a priori estimates in the norm of X_3 is sufficient by (2.12), i.e.,

$$|||\theta(t)|||_3^2 \equiv \|\theta(t)\|_{H^3}^2 + \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}\|_{H^1}^2 < +\infty \tag{2.16}$$

for $t \geqslant 0$.

For establishing the L^2-estimates on $\theta(t)$, we need a sequence of lemmas. Define $Q_T = R \times [0, T]$, for any $T > 0$, we introduce the transformation

$$\eta = \frac{x}{\sqrt{t+1}}, \qquad \tau = \log(1+t)$$

under which Q_T becomes into $S_{T'} = R \times [0, T']$ on the plane (η, τ) where $T' = \log(1+T)$. Let us still use the notation $\tilde{v}(\eta, \tau)$ to denote the solution $\tilde{v}(x, t)$ of (2.2), it follows that

$$\tilde{v}_\tau = [-p(\tilde{v}, s)]_{\eta\eta} + \tfrac{1}{2}\eta\tilde{v}_\eta, \qquad \tau \in [0, T'] \tag{2.17}$$

$$\tilde{v}(\eta, 0) = e^{(1/\gamma)(s(x) - \bar{s})} v^*(x + x_0), \tag{2.18}$$

where $s = s(x) = s(\eta e^{\tau/2})$.

Lemma 2.1. *Under the assumptions* (1.7), (1.9), *and* (2.14) *on* $s_0(x)$, *there are constants* $\eta_0 \geqslant N$, $\eta_1 \leqslant -N$, $\beta_i > 0$ *and* $m_i > 1$ $(i = 1, 2)$ *such that for any positive smooth solution of* (2.17)–(2.18) *it holds that*

$$\underline{h}(\eta, \tau) \leqslant \tilde{v}(\eta, \tau) \leqslant \bar{h}(\eta, \tau), \tag{2.19}$$

where the continuous functions $\underline{h}$ and $\bar{h}$ *are defined as follows*

$$\underline{h}(\eta,\tau)=\begin{cases} e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}v^-, & \eta\leqslant\eta_0\\ \max\{e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}v^-,\, v^+-\beta_1(\eta-\eta_0)^{-m_1}\} \\ =\max\{v^-,\, v^+-\beta_1(\eta-\eta_0)^{-m_1}\}, & \eta>\eta_0\end{cases}$$

and

$$\bar{h}(\eta,\tau)=\begin{cases} e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}v^+, & \eta\geqslant\eta_1\\ \min\{e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}v^+,\, v^-+\beta_2(-\eta+\eta_1)^{-m_2}\} \\ =\min\{v^+,\, v^-+\beta_2(\eta-\eta_1)^{-m_2}\}, & \eta<\eta_1.\end{cases}$$

Proof. Due to (2.5), (2.7) and (2.8), it is easy to check that the functions

$$e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}\cdot v^- \qquad \text{and} \qquad e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}\cdot v^+$$

are lower and upper solutions for (2.17)–(2.18) respectively. Thus, the Maximum principle implies that

$$e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}\cdot v^- \leqslant \tilde{v}(\eta,\tau)\leqslant e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}\cdot v^+ \tag{2.20}$$

for $(\eta,\tau)\in S_{T'}$.

We show the left half of (2.19) first. For the initial data $\tilde{v}(\eta,0)$, it can be claimed, due to (2.4), that there exist constants $\eta_0\geqslant N$, $\beta_1>0$ and $m_1>1$ such that $\underline{h}(\eta,0)\leqslant\tilde{v}(\eta,0)$.

Choose $\hat{\eta}>\eta_0$ by $v^+-\beta_1(\hat{\eta}-\eta_0)^{-m_1}=v^-$, namely, $\hat{\eta}=\eta_0+((v^+-v^-)/\beta_1)^{-1/m_1}$. It can be checked then that

$$\underline{h}(\eta,\tau)=\begin{cases} e^{(1/\gamma)[s(\eta e^{\tau/2})-\bar{s}]}\cdot v^-, & \eta\leqslant\hat{\eta}\\ v^+-\beta_1(\eta-\eta_0)^{-m_1}, & \eta\geqslant\hat{\eta}.\end{cases}$$

This implies immediately, with the help of (2.20), that $\underline{h}(\eta,\tau)\leqslant\tilde{v}(\eta,\tau)$ for $\eta\leqslant\hat{\eta}$, $\tau\in[0,\log(1+T)]$. Turn to $\eta\geqslant\hat{\eta}$ now. At $\eta=\hat{\eta}$, $\underline{h}(\hat{\eta},\tau)=v^-\leqslant\tilde{v}(\hat{\eta},\tau)$, $\tau\in[0,\log(1+T)]$. Denote $\mu(\eta)=v^+-\beta_1(\eta-\eta_0)^{-m_1}$. It is sufficient to verify that

$$\mu_\tau+p(\mu,\bar{s})_{\eta\eta}-\tfrac{1}{2}\eta\mu_\eta\leqslant 0 \qquad \text{for}\quad \eta\geqslant\hat{\eta},\quad \tau\in[0,\log(1+T)], \tag{2.21}$$

since $s(\eta e^{\tau/2})\equiv\bar{s}$ for $\eta\geqslant\hat{\eta}>\eta_0\geqslant N$.

Due to the fact that $0<v^-\leqslant\mu(\eta)\leqslant v^+(\eta\geqslant\hat{\eta})$, there exists a constant $\lambda_0>0$ such that

$$p_\mu(\mu,\bar{s})\geqslant-\lambda_0,$$

and $-p_{\mu\mu}(\mu,s)\geqslant-\lambda_0$.

Thus, it is claimed that (2.21) can be guaranteed by

$$\tfrac{1}{2}\eta(\eta-\eta_0)^{m_1+1}-\lambda_0(m_1+1)(\eta-\eta_0)^{m_1}-\lambda_0 m_1\beta_1 \geqslant 0, \qquad (\eta \geqslant \hat{\eta})$$

which is true if we choose η_0 such that

$$\eta_0 \geqslant \max\left\{N, 2\left[\frac{v^+-v^-}{\beta_1}\right]^{1/m_1}[\lambda_0(m_1+1)+\lambda_0 m_1(v^+-v^-)]\right\}.$$

Using the same argument as in [Co, Lemma 1] for the domain $\{(\eta,\tau)\colon \eta\geqslant\hat{\eta},\ \tau\in[0,\log(1+T)]\}$, we arrive at $\underline{h}(\eta,\tau)\leqslant\tilde{v}(\eta,\tau)$, $\eta\geqslant\hat{\eta}$, $\tau\in[0,\log(1+T)]$. This finishes the proof for the left half of (2.19). The right half can be proved similarly.

We compare a positive smooth solution $\tilde{v}(x,t)$ with $v^*((x+x_0+a_0)/\sqrt{t+1})$ in the next lemma.

LEMMA 2.2. *For any positive smooth solution of* (2.2) *with* $\tilde{v}_x(\cdot,t)\in H^1(R)(t\in[0,T])$, *it holds that*

$$\begin{aligned}\int_{-\infty}^{\infty}&\left|v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right|dx\\ &\leqslant\Big[4k_0\sup_{x\in[-N,N]}e^{(1/\gamma)(s(x)-\bar{s})}+a_0\Big](v^+-v^-)\\ &\quad+(4k_0+2N)\,v^+\cdot\sup_{x\in[-N,N]}|e^{(1/\gamma)(s(x)-\bar{s})}-1|,\end{aligned}\tag{2.22}$$

where k_0 *is the constant in* (1.17).

Proof. Let us still use the notation $v^*(\eta,\tau)$ to denote $v^*((x+x_0+a_0)/\sqrt{t+1})$, namely, $v^*(\eta,\tau)=v^*(\eta+(x_0+a_0)\,e^{-\tau/2})$ which satisfies

$$\begin{aligned}v^*_\tau&=[-p(v^*,\bar{s})]_{\eta\eta}+\tfrac{1}{2}\eta v^*_\eta\\ v^*(\eta,0)&=v^*(x+x_0+a_0).\end{aligned}\tag{2.23}$$

In view of (2.4) and (2.19), it can be shown that the function $\Phi(\tau)$

$$\Phi(\tau)=\int_{-\infty}^{\infty}[v^*(\eta+(x_0+a_0)\,e^{-\tau/2})-\tilde{v}(\eta,\tau)]\,dn$$

is well defined for any $\tau\in[0,T']$.

Subtracting (2.17) from (2.23) and integrating it over $(-\infty,\infty)$, with the help of integration by parts and the facts that $v^*_\eta\to 0$ as $\eta\to\mp\infty$

(see [DP1, DP2, AP1, AP2]), $\tilde{v}_\eta \to 0$ as $\eta \to \mp\infty$, and $s(\eta e^{\tau/2}) \equiv \bar{s}$ for $|\eta| \geqslant N$, we obtain

$$\frac{d\Phi}{d\tau} = \frac{1}{2}\eta(v^* - \tilde{v})\Big|_{-\infty}^{\infty} - \frac{1}{2}\Phi(\tau) \qquad \text{for} \quad \tau \in [0, T']$$

which, with (2.4) and (2.19) together, implies

$$\Phi(\tau) = \Phi(0)\, e^{-\tau/2} \qquad \text{for} \quad \tau \in [0, T'] \tag{2.24}$$

where

$$\Phi(0) = \int_{-\infty}^{\infty} [v^*(x + x_0 + a_0) - e^{(1/\gamma)(s(x) - \bar{s})} v^*(x + x_0)]\, dx. \tag{2.25}$$

Subtracting the equation satisfied by $v^*(x + x_0)$ from the one by $v^*(x + x_0 + a_0)$ and integrating it over $(-\infty, \infty)$, it turns out that

$$\int_{\infty}^{\infty} [v^*(x + x_0 + a_0) - v^*(x + x_0)]\, dx = a_0(v^+ - v^-) \tag{2.26}$$

with the help of (2.4), the fact of $v^*_x \to 0$ as $x \to \mp\infty$ and integration by parts. This, together with $0 < v^- \leqslant v^* \leqslant v^+$ and support $\{s(x) - \bar{s}\} \subset [-N, N]$, yields

$$\Phi(0) \leqslant a_0(v^+ - v^-) + 2Nv^+ \cdot \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| \tag{2.27}$$

which, combined with (2.24), implies

$$\int_{-\infty}^{\infty} \left[v^* \left(\frac{x + x_0 + a_0}{\sqrt{t+1}} \right) - \tilde{v}(x, t) \right] dx$$
$$\leqslant a_0(v^+ - v^-) + 2Nv^+ \cdot \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| \tag{2.28}$$

for $t \in [0, T]$

Due to condition V, it is known that

$$\int_{-\infty}^{+\infty} \left| v^* \left(\frac{x + x_0 + a_0}{\sqrt{t+1}} \right) - \tilde{v}(x, t) \right| dx$$
$$= \int_{-\infty}^{+\infty} \left[v^* \left(\frac{x + x_0 + a_0}{\sqrt{t+1}} \right) - \tilde{v}(x, t) \right] dx$$

$$+\int_{-k_0}^{k_0}\left\{\left|v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right|\right.$$

$$\left.-\left[v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right]\right\}dx$$

$$\leqslant\int_{-\infty}^{+\infty}\left[v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right]dx$$

$$+2\int_{-k_0}^{k_0}\left|v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right|dx, \tag{2.29}$$

where

$$\int_{-k_0}^{k_0}\left|v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right|dx$$

$$\leqslant\int_{-k_0}^{k_0}\left|v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-e^{(1/\gamma)(s-\bar{s})}v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)\right|dx$$

$$+\int_{-k_0}^{k_0}\left|e^{(1/\gamma)(s-\bar{s})}v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right|dx. \tag{2.30}$$

In view of (2.5) and (2.20), it can be shown that

$$\left|e^{(1/\gamma)(s-\bar{s})}v^*\left(\frac{x+x_0+a_0}{\sqrt{t+1}}\right)-\tilde{v}(x,t)\right|\leqslant e^{(1/\gamma)(s-\bar{s})}(v^+-v^-) \tag{2.31}$$

(2.22) follows then from (2.28)–(2.31).

For establishing the a priori estimates needed, we collect more information about the similarity solution $v^*=v^*((x+x_0+a_0)/\sqrt{t+1})$ of (1.16)

LEMMA 2.3. *The following estimates hold for* $v^*=v^*((x+x_0+a_0)/\sqrt{t+1})$

$$\|v_x^*\|_{H^2}+\|v_t^*\|_{H^2}+\|v_{tt}^*\|_{H^1}+|v_x^*|_{c^1}+|v_t^*|_{c^1}+|v_{tt}^*|_{c^0}\leqslant c\,|v_\eta^*(0)| \tag{2.32}$$

$$\int_{-\infty}^{+\infty}(v_t^*)^2(x,t)\,dx\leqslant\frac{c[v_\eta^*(0)]^2}{(t+1)^{3/2}}$$

$$\int_{-\infty}^{+\infty}(v_{tx}^*)^2(x,t)\,dx\leqslant\frac{c[v_\eta^*(0)]^2}{(t+1)^{5/2}}$$

$$\int_{-\infty}^{+\infty}[(v_{tt}^*)^2+(v_{txx}^*)^2](x,t)\,dx\leqslant\frac{c[v_\eta^*(0)]^2}{(t+1)^{7/2}}$$

$$\int_{-\infty}^{+\infty} [(v^*_{ttx})^2 + (v^*_{txxx})^2](x,t)\,dx \leqslant \frac{c[v^*_\eta(0)]^2}{(t+1)^{9/2}}$$

$$\int_{-\infty}^{+\infty} (v^*_{ttxx})^2\,(x,t)\,dx \leqslant \frac{c[v^*_\eta(0)]^2}{(t+1)^{11/2}} \tag{2.33}$$

$$\int_{-\infty}^{+\infty} (v^*_{ttt})^2\,(x,t)\,dx \leqslant \frac{c[v^*_\eta(0)]^2}{(t+1)^{11/2}} \tag{2.34}$$

$$\int_{-\infty}^{+\infty} (v^*_{tttx})^2\,(x,t)\,dx \leqslant \frac{c[v^*_\eta(0)]^2}{(t+1)^{13/2}}. \tag{2.35}$$

Moreover,

$$|v^*_{xxx}|_{c^0} \leqslant c\,|v^*_\eta(0)| \tag{2.36}$$

and

$$|v^*_\eta(\eta)| \leqslant c\,|v^+ - v^-|^{1/3}, \tag{2.37}$$

where c is a positive constant depending only on v^+, v^-, $\bar{s}$, and γ.

Proof. All of the estimates cited in Lemma 2.3, except (3.36) and (3.37), can be obtained in the same way as in [HLI1] and [HLI2].

Differentiating the equation (1.16) with respect to x and expressing v^*_{xxx} in terms of v^*_{xt}, v^*_{xx} and v^*_x, we arrive at (2.36), with the help of (2.5) and (2.32).

Due to $v^*_\eta > 0$ (since $v^- < v^+$) and $v^*_\eta(\eta) \to 0$ as $\eta \to \mp\infty$, it holds that

$$[v^*_\eta(\eta)]^2 \leqslant 2[\sup_{\eta\in(-\infty,\infty)} v^*_\eta(\eta)]^{1/2} \left[\int_{-\infty}^{+\infty} v^*_\eta(\eta)\,d\eta\right]^{1/2} \times \left\{\int_{-\infty}^{+\infty} [v^*_{\eta\eta}(\eta)]^2\,d\eta\right\}^{1/2}.$$

This, together with the estimate $\int_{-\infty}^{+\infty} [v^*_{\eta\eta}(\eta)]^2\,d\eta \leqslant \hat{A}_0$ (see [HLI1] and [HLI2], $\hat{A}_0$ is a positive constant depending only on v^+, v^-, $\bar{s}$, and γ), yields (2.37).

Denote

$$a_1 = \inf_{x\in(-\infty,\infty)} e^{(1/\gamma)(s(x)-\bar{s})} v^-$$

$$a_2 = \sup_{x\in(-\infty,\infty)} e^{(1/\gamma)(s(x)-\bar{s})} v^+$$

$$\alpha_0 = \inf_{\substack{v\in[a_1,a_2]\\ s\in[\inf_R s_0(x),\,\sup_R s_0(x)]}} (-p_v(v,s)) > 0$$

$$M = \sup_{\substack{v\in[a_1,a_2]\\ s\in[\inf_R s_0(x),\,\sup_R s_0(x)]}} \sum_{i=0}^{5}\sum_{l=0}^{i}\left|\frac{\partial^i p(v,s)}{\partial v^{(l)}\,\partial s^{(i-l)}}\right|.$$

For fixed $\hat{l}$: $0<\hat{l}<\min\{1, a_1\}$, define $\Omega := \{\theta\colon |\theta| < \hat{l}\}$, we will seek a smooth solution $\theta\in\Omega$ for the Cauchy problem (2.9)–(2.10) in $t\in[0,T]$. A positive smooth solution $\tilde{v}$ of (2.2) can be obtained then by $\tilde{v} := \hat{v}+\theta > 0$ for which it holds

$$a_1 \leqslant \tilde{v} = \hat{v} + \theta \leqslant a_2 \tag{2.38}$$

from (2.20).

Introduce

$$e(t) = \sum_{j=1}^{3} e_j(t) \tag{2.39}$$

for the solution $\theta(x,t)$ with $\theta\in\Omega$ in each $(x,t)\in R\times[0,T]$, where

$$e_1(t) = \int_{-\infty}^{+\infty}\left[\theta^2(x,t) + \frac{\theta_x^2(x,t)}{2} + \frac{\theta_t^2(x,t)}{2}\right]dx$$

$$e_2(t) = \frac{1}{2}\int_{-\infty}^{+\infty}[\theta_{tt}^2(t,x) + (1+\alpha_0)\,\theta_{xt}^2(x,t)]\,dx$$

$$e_3(t) = \frac{1}{2}\int_{-\infty}^{+\infty}[\theta_{xxt}^2(x,t) + \theta_{ttx}^2(x,t)]\,dx.$$

Due to (2.9), (2.10), Lemma 2.3, (2.12), and (2.38), it follows that

$$\|\theta(t)\|_2^2 \leqslant K\,|||\theta(t)|||_3^2 \leqslant \bar{K}e(t) + \bar{R}[v_\eta^*(0)]^2 + \bar{L}\,\|s(x)-\bar{s}\|_{H^5}^2 \tag{2.40}$$

provided $\|s(x)-\bar{s}\|_{H^5} = \delta_0 \leqslant 1$ and $|v_\eta^*(0)| \leqslant 1$, where $\bar{K}$, $\bar{R}$, and $\bar{L}$ depend only on a_1, a_2, α_0, M, and b.

Lemma 2.4. *Under the assumptions* (2.14), (1.7), *and* (1.9), *if the solution* $\theta(t)\in X_3$ $(t\in[0,T])$ *to the Cauchy problem* (2.9)–(2.10) *satisfies*

$$\|\theta(t)\|_2 < \hat{l}, \qquad 0\leqslant t\leqslant T \tag{2.41}$$

and it holds that

$$\begin{aligned} &\|s(x)-\bar{s}\|_{H^5} \leqslant 1, && |v_\eta^*(0)| \leqslant 1,\\ &\sup|e^{(1/\gamma)(s(x)-\bar{s})} - 1| \leqslant 1, && |v^+ - v^-| \leqslant 1 \end{aligned} \tag{2.42}$$

then one has the a priori estimate

$$e(t) \leqslant K_1 e(0) + R_1 [v_\eta^*(0)]^2 + L_1 \{\sup |e^{(1/\gamma)(s(x)-\bar{s})} - 1| + \|s(x) - \bar{s}\|_{H^5}^2\} + D_1 |v^+ - v^-|, \tag{2.43}$$

$t \in [0, T]$, *where* K_1, R_1, L_1, *and* D_1 *are constants depending only on* a_1, a_2, α_0, M, b, N, k_0, a_0, *and* $v^{\mp}$.

For proving, we first assume that the solution $\theta(t)$ belongs to the space X_4. The estimate (2.43) is also valid for the solution $\theta(t) \in X_3$ by use of the Friedricgh's mollifier under the same assumptions (2.41) and (2.42) which we omit. The constants K_i, R_i, L_i, and D_i $(i = 2, ..., 8)$, used in the following proof, depend on the same quantities as K_1, R_1, L_1, and D_1 do.

Proof. Due to (2.38) and Lemma 2.2, it can be shown that

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\theta(x, t)| \, dx
&\leqslant \int_{-\infty}^{+\infty} \left| [e^{(1/\gamma)(s(x)-\bar{s})} - 1] \, v^* \left(\frac{x + x_0 + a_0}{\sqrt{t+1}} \right) \right| dx \\
&\quad + \int_{-\infty}^{+\infty} \left| v^* \left(\frac{x + x_0 + a_0}{\sqrt{t+1}} \right) - \tilde{v}(x, t) \right| dx \\
&\leqslant L_2 \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| + D_2 |v^+ - v^-|, \qquad t \in [0, T].
\end{aligned}$$

This, together with (2.41), implies

$$\int_{-\infty}^{+\infty} \theta^2(x, t) \, dx \leqslant L_2 \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| + D_2 |v^+ - v^-|, \qquad t \in [0, T]. \tag{2.44}$$

Using (1.16) and (2.8), the equation (2.9) can be rewritten as

$$\theta_t = -p(\hat{v} + \theta, s)_{xx} + p(\hat{v}, s)_{xx} + [1 - e^{(1/\gamma)(s(x)-\bar{s})}] v_t^*. \tag{2.45}$$

Let

$$q(x, t) = [-p(\hat{v} + \theta, s) + p(\hat{v}, s)](x, t). \tag{2.46}$$

Then

$$\theta_t = q_{xx} + [1 - e^{(1/\gamma)(s(x)-\bar{s})}] v_t^*. \tag{2.47}$$

Hereafter $v^* = v^*((x + x_0 + a_0)/\sqrt{t+1})$.

Define $U=(\partial/\partial x)[q(x,t)]$, it follows that

$$U_t = q_{xt}.$$

Multiplying the above equation by U and integrating it over $Q_t := R\times[0,t]$, with the help of integration by parts, using Lemma 2.3, (2.46)–(2.47), and Young's inequality, we arrive at

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{U^2}{2}(x,t)\,dx+\frac{\alpha_0}{2}\int_0^t\int_{-\infty}^{+\infty}\theta_t^2\,dx\,d\tau\\ \leqslant\int_{-\infty}^{+\infty}\frac{U^2}{2}(x,0)\,dx+R_0[v_\eta^*(0)]^2,\end{aligned}\tag{2.48}$$

where R_0 is a constant depending only on the quantities cited in Lemma 2.4.

In view of (2.5), (2.7), (2.38), (2.40), (2.46), (2.48), and Lemma 2.3, it is known that

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{+\infty}\theta_x^2(x,t)\,dx+\frac{\alpha_0}{2}\int_0^t\int_{-\infty}^{+\infty}\theta_t^2(x,\tau)\,dx\,d\tau\\ \leqslant K_3e(0)+R_3[v_\eta^*(0)]^2+L_3\,\|s(x)-\bar{s}\|_{H^5}^2.\end{aligned}\tag{2.49}$$

Differentiate (2.9) with respect to t and multiply the equation by θ_t then, one obtains the following inequality by integration and using Lemma 2.3, (2.49), (2.41), (2.42) and Young's inequality,

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{+\infty}\theta_t^2(x,t)\,dx+\frac{\alpha_0}{2}\int_0^t\int_{-\infty}^{+\infty}\theta_{xt}^2(x,\tau)\,dx\,d\tau\\ \leqslant K_4e(0)+R_4[v_\eta^*(0)]^2+L^4\,\|s(x)-\bar{s}\|_{H^5}^2.\end{aligned}\tag{2.50}$$

Differentiate (2.9) with respect to t and multiply the resulting equation by θ_{tt}, integrate it over $[0,t]\times(-\infty,\infty)$ then, we get

$$\begin{aligned}\frac{\alpha_0}{2}\int_{-\infty}^{+\infty}\theta_{xt}^2(x,t)\,dx+\frac{1}{2}\int_0^t\int_{-\infty}^{+\infty}\theta_{tt}^2(x,\tau)\,dx\,d\tau\\ \leqslant K_5e(0)+R_5[v_\eta^*(0)]^2+L_5\,\|s(x)-\bar{s}\|_{H^5}^2\end{aligned}\tag{2.51}$$

with the help of integration by parts and Lemma 2.3, (2.41), (2.42), (2.49), (2.50), and Young's inequality.

Differentiate (2.9) with respect to t and x successively and multiply the resulting equation by θ_{xt}, integrate it over $[0,t]\times(-\infty,+\infty)$ then, it becomes, by (2.49)–(2.51), (2.41), (]2.42), Young's inequality, and Lemma 2.3, that

$$\int_{-\infty}^{+\infty} \frac{1}{2}\,\theta_{xt}^2(x,t)\,dx + \frac{\alpha_0}{2}\int_0^t \int_{-\infty}^{+\infty} \theta_{xxt}^2(x,\tau)\,dx\,d\tau$$

$$\leqslant K_6 e(0) + R_6[v_\eta^*(0)]^2 + L_6\,\|s(x)-\bar{s}\|_{H^5}^2. \tag{2.52}$$

Differentiate (2.9) with respect to t twice and multiply the equation by θ_{tt}, integrate it over $[0, t]\times(-\infty, \infty)$ then, it follows, from (2.41), (2.42), (2.49)–(2.52), Lemma 2.3, and Young's inequality, that

$$\int_{-\infty}^{+\infty} \frac{\theta_{tt}^2}{2}(x,t)\,dx + \frac{\alpha_0}{2}\int_0^t \int_{-\infty}^{+\infty} \theta_{xtt}^2(x,\tau)\,dx\,d\tau$$

$$\leqslant K_7 e(0) + R_7[v_\eta^*(0)]^2 + L_7\,\|s(x)-\bar{s}\|_{H^5}^2. \tag{2.53}$$

With a similar approach as in [HS] and above, we differentiate (2.9) with respect to t twice and x once successively, multiply it by θ_{ttx} and integrate the equation over $[0, t]\times(-\infty, \infty)$, we obtain then an estimate concerning $\int_{-\infty}^{+\infty} (\theta_{ttx}^2/2)(x,t)\,dx$ and $\int_0^t \int_{-\infty}^{+\infty} \theta_{ttxx}^2(x,\tau)\,dx\,d\tau$.

Differentiate (2.9) with respect to t once and x twice successively, multiply it by θ_{xxt} and integrate the equation over $[0, t]\times(-\infty, \infty)$, we get an estimate on $\int_{-\infty}^{+\infty} (\theta_{xxt}^2/2)(x,t)\,dx$ and $\int_0^t \int_{-\infty}^{+\infty} \theta_{xxxt}^2(x,\tau)\,dx\,d\tau$.

These two kind of estimates, mentioned above, yield

$$\int_{-\infty}^{+\infty} \left[\frac{\theta_{xxt}^2}{2} + \frac{\theta_{ttx}^2}{2}\right](x,t)\,dx + \frac{\alpha_0}{2}\int_0^t \int_{-\infty}^{+\infty} [\theta_{xxxt}^2 + \theta_{xxtt}^2](x,\tau)\,dx\,d\tau$$

$$\leqslant K_8 e(0) + R_8[v_\eta^*(0)]^2 + L_8\,\|(s(x)-\bar{s})\|_{H^5}^2. \tag{2.54}$$

The lemma follows then from (2.44), (2.49)–(2.54).

As far as $e(0)$ is concerned, it is not difficult to show, by using the properties of similarity solutions, obtained in [HLI1] and [HLI2], that

$$e(0) \leqslant \hat{K}\,\|(s(x)-\bar{s})\|_{H^5}^2 + \hat{R}[v_\eta^*(0)]^2,$$

where $\hat{K}$ and $\hat{R}$ depend only on those quantities cited in Lemma 2.4.

LEMMA 2.5. *Assume* $|v^+ - v^-|$ *and* $\|s(x)-\bar{s}\|_{H^5}$ *are suitably small so that*

$$2\bar{K}e(0) + \bar{R}[v_\eta^*(0)]^2 + \bar{L}\,\|s(x)-\bar{s}\|_{H^5}^2 < \hat{l}^2,$$

and

$$
\begin{aligned}
&2\bar{K}K_1 e(0) + (2R_1 + \bar{R})[v_\eta^*(0)]^2 + (2L_1 + \bar{L}) \|s(x) - \bar{s}\|_{H^5}^2 \\
&\quad + 2(L_1 \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| + D_1 |v^+ - v^-|) < \hat{l}^2.
\end{aligned}
$$

Then there exists a unique globally defined smooth solution $\theta \in X_3$ *for Cauchy problem* (2.9)–(2.10). *Where* $\bar{K}$, $\bar{R}$ *and* $\bar{L}$ *are the constants in* (2.40); K_1, R_1, L_1 *and* D_1 *are the constants in Lemma* 2.4; $s(x) = s_0(x)$ *satisfies* (1.7) *and* (1.9).

Proof. By the local existence theorem there exists $t_1 > 0$ such that the solution $\theta(t) \in X_3$ exists in $0 \leqslant t \leqslant t_1$ and satisfies

$$e(t) \leqslant 2e(0) \qquad \text{in} \quad 0 \leqslant t \leqslant t_1.$$

This, with (2.40) together, yields

$$
\begin{aligned}
\|\theta(t)\|_2^2 &\leqslant \bar{K}e(t) + \bar{R}[v_\eta^*(0)]^2 + \bar{L} \|s(x) - \bar{s}\|_{H^5}^2 \\
&\leqslant 2\bar{K}e(0) + \bar{R}[v_\eta^*(0)]^2 + \bar{L} \|s(x) - \bar{s}\|_{H^5}^2 \\
&< \hat{l}^2 \qquad \text{in} \quad 0 \leqslant t \leqslant t_1.
\end{aligned}
$$

Thus, Lemma 2.4 implies

$$
\begin{aligned}
e(t) &\leqslant K_1 e(0) + R_1[v_\eta^*(0)]^2 \\
&\quad + L_1[\sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| + \|s(x) - \bar{s}\|_{H^5}^2] \\
&\quad + D_1 |v^+ - v^-| \qquad \text{in} \quad 0 \leqslant t \leqslant t_1.
\end{aligned} \tag{2.55}
$$

Next, by the local existence theorem for $t \geqslant t_1$, there exists a $\tilde{t} > 0$ such that the solution $\theta(t) \in X_3$ exists in $0 \leqslant t \leqslant t_1 + \tilde{t}$ and satisfies

$$e(t) \leqslant 2e(t_1) \qquad \text{in} \quad t_1 \leqslant t \leqslant t_1 + \tilde{t}.$$

This together with (2.40) and (2.55) yields

$$
\begin{aligned}
\|\theta(t)\|_2^2 &\leqslant 2\bar{K}K_1 e(0) + (2R_1 + \bar{R})[v_\eta^*(0)]^2 + (2L_1 + \bar{L}) \|s(x) - \bar{s}\|_{H^5}^2 \\
&\quad + 2\{L_1 \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| + D_1(v^+ - v^-)\} \\
&< \hat{l}^2 \qquad \text{in} \quad t_1 \leqslant t \leqslant t_1 + \tilde{t}.
\end{aligned}
$$

Thus, Lemma 2.4 implies

$$
\begin{aligned}
e(t) \leqslant K_1 e(0) &+ R_1[v_\eta^*(0)]^2 \\
&+ L_1\Big[\sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})}-1| + \|s(x)-\bar{s}\|_{H^5}^2\Big] \\
&+ D_1\,|v^+ - v^-| \qquad \text{in} \quad 0\leqslant t\leqslant t_1+\tilde{t}.
\end{aligned}
$$

Repeat the same procedure with the same time interval $\tilde{t}>0$, we complete the proof.

LEMMA 2.6. *Under the same assumptions as in Lemma 2.5, the function $\tilde{v}(x,t)$, defined by $\tilde{v}(x,t)=\hat{v}(x,t)+\theta(x,t)$, is the unique positive smooth solution of Cauchy problem (2.2) for which it holds that*

$$
\begin{aligned}
&\|\tilde{v}_x\|_{H^2}^2 + \|\tilde{v}_t\|_{H^2}^2 + \|\tilde{v}_{tt}\|_{H^1}^2 \\
&\qquad + \int_0^t\int_{-\infty}^{+\infty} [\tilde{v}_t^2 + \tilde{v}_{xt}^2 + \tilde{v}_{tt}^2 + \tilde{v}_{xxt}^2 + \tilde{v}_{ttx}^2](x,\tau)\,dx\,d\tau \\
&\quad \leqslant B_1\{\|s(x)-\bar{s}\|_{H^5}^2 + \sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})}-1| + [v_\eta^*(0)]^2 + |v^+-v^-|\}
\end{aligned}
\tag{2.56}
$$

$$
\begin{aligned}
&|\tilde{v}_x|_{C^1}^2 + |\tilde{v}_t|_{C^1}^2 + |\tilde{v}_{tt}|_{C^0}^2 \\
&\quad \leqslant B_2\{\|s(x)-\bar{s}\|_{H^5}^2 + \sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})}-1| + [v_\eta^*(0)]^2 + |v^+-v^-|\}.
\end{aligned}
\tag{2.57}
$$

for any $t\geqslant 0$.

Moreover, the same kind of estimate is true for $|\tilde{v}_{xxx}^2|_{C^0}$ and it holds that

$$
a_1 \leqslant e^{(1/\gamma)(s(x)-\bar{s})}v^- \leqslant \tilde{v}(x,t) \leqslant e^{(1/\gamma)(s(x)-\bar{s})}v^+ \leqslant a_2. \tag{2.58}
$$

Furthermore, if $\theta(t)\in X_4$, the estimate, similar to (2.56) *and* (2.57), *holds for $\int_0^t\int_{-\infty}^{+\infty} (\tilde{v}_{xxxt}^2 + \tilde{v}_{xxtt}^2)(x,\tau)\,dx\,d\tau$.*

The constants B_1 and B_2 depend only on the quantities cited in Lemma 2.4.

Proof. All of the estimates except $|\tilde{v}_{xxx}^2|_{C^0}$ can be obtained directly from Lemma 2.1, Lemmas 2.3–2.5 and (2.12). It is easy to see that $\tilde{v}_{xxx}$ can be expressed in terms of $\tilde{v}_{xt}$, $\tilde{v}_{xx}$, $\tilde{v}_x$, s_{xxx}, s_{xx}, and s_x. Thus, (2.57) and (2. 58) yield the required estimate of $\tilde{v}_{xxx}$.

We have known from Lemma 2.5 that $\tilde{v}(x,t)=\hat{v}(x,t)+\theta(x,t)$ is a globally defined solution of (2.2) satisfying $\tilde{v}(x,t)>0$. Due to Maximum principle, the globally defined positive smooth solution of (2.2) is unique. Therefore, this solution is just $\tilde{v}(x,t)=\hat{v}(x,t)+\theta(x,t)$.

We turn to the Cauchy problem (2.1) next.

Let $\psi = v(x, t) - \tilde{v}(x, t)$, $t \in [0, T]$. Where $v(x, t) > 0$ is a smooth solution of (2.1). We get the following Cauchy problem for $\psi(x, t)$.

$$\begin{aligned} \psi_t &= -p(\tilde{v}+\psi, s)_{xx} + p(\tilde{v}, s)_{xx} \\ \psi(x, 0) &= e^{(1/\gamma)(s(x)-\bar{s})}[v^*(x+x^*) - v^*(x+x_0)]. \end{aligned} \tag{2.59}$$

The global existence for (2.59) can be investigated in X_3, similarly to the Cauchy problem (2.9)–(2.10).

Define

$$\hat{e}(t) = \sum_{j=1}^{3} \hat{e}_j(t) \tag{2.60}$$

for any solution $\psi(t) \in X_3$ with $\|\psi(t)\|_2 < \hat{l}$, where $\hat{e}_j$ $(j = 1, 2, 3)$ is defined by the formula in (2.39) with ψ instead of θ. In view of Lemma 2.3, Lemma 2.6, (2.12), and (2.59), it can be shown that

$$\begin{aligned} \|\psi(t)\|_2^2 \leqslant K \,|||\psi(t)|||_3^2 \leqslant A[\hat{e}(t) + \|s(x) - \bar{s}\|_{H^5}^2 \\ + |v^+ - v^-| + \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1|], \end{aligned} \tag{2.61}$$

where A is a positive constant, depending only on the quantities cited in Lemma 2.4.

Similar to the estimate on $e(0)$, noting that $|v^*(x+x^*) - v^*(x+x_0)| \leqslant v^+ - v^-$, we obtain, with the help of the corresponding estimates on the similarity solution in [HLI1] and [HLI2], that

$$\begin{aligned} \hat{e}(0) \leqslant \hat{B}\{[v_\eta^*(0)]^2 + \|s(x) - \bar{s}\|_{H^5}^2 \\ + \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| + |v^+ - v^-|\}, \end{aligned} \tag{2.62}$$

where $\hat{B}$ depends only on the quantities cited in Lemma 2.4.

LEMMA 2.7. *Under the assumptions introduced in Lemma 2.5, if the solution $\psi(t) \in X_3(t \in [0, T])$ to the Cauchy problem (2.59) satisfies*

$$\|\psi(t)\|_2 < \hat{l}, \qquad 0 \leqslant t \leqslant T \tag{2.63}$$

and it holds that

$$\hat{e}(0) + [v_\eta^*(0)]^2 + \|s(x) - \bar{s}\|_{H^5}^2 + \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| + |v^+ - v^-| < 1,$$

then one has the a priori estimate

$$\hat{e}(t) \leqslant K_4\{\hat{e}(0) + [v_\eta^*(0)]^2 + \|s(x)-\bar{s}\|_{H^5}^2 + \sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| + |v^+ - v^-|\}^{1/3} \qquad t\in[0,T]. \tag{2.64}$$

For the proof of Lemma 2.7, we need the following Lemma 2.8. By the translation $\eta = (x/\sqrt{t+1})$, $\tau = \log(1+t)$, (2.1) and (2.2) become

$$\begin{aligned} v_\tau &= -p(v,s)_{\eta\eta} + \tfrac{1}{2}\eta v_\eta \\ v(\eta,0) &= e^{(1/\gamma)(s(x)-\bar{s})} v^*(x+x^*) \end{aligned} \tag{2.65}$$

and

$$\begin{aligned} \tilde{v}_\tau &= -p(\tilde{v},s)_{\eta\eta} + \tfrac{1}{2}\eta \tilde{v}_\eta \\ \tilde{v}(\eta,0) &= e^{(1/\gamma)(s(x)-\bar{s})} v^*(x+x_0). \end{aligned} \tag{2.66}$$

Similar to Lemma 2.1, we have

LEMMA 2.8. *Under the assumptions* (2.14) *on* $s_0(x)$, *there are constants* $\eta'_0 \geqslant N$, $\eta'_1 \leqslant -N$, $\beta'_i > 0$ *and* $m'_i > 1$ $(i=1,2)$ *such that for any positive smooth solution* $v(\eta,\tau) > 0$ *of* (2.65) $((\eta,\tau)\in S_{T'})$ *it holds that*

$$\underline{f}(\eta,\tau) \leqslant v(\eta,\tau) \leqslant \bar{f}(\eta,\tau), \tag{2.67}$$

where the continuous functions $\underline{f}$ *and* $\bar{f}$ *are defined as follows*

$$\underline{f}(\eta,\tau) = \begin{cases} e^{(1/\gamma)(s(\eta e^{\tau/2})-\bar{s})} v^- & \eta \leqslant \eta'_0 \\ \max\{e^{(1/\gamma)(s(\eta e^{\tau/2})-\bar{s})} v^-, v^+ - \beta'_1(\eta-\eta'_0)^{-m'_1}\} & \\ = \max\{v^-, v^+ - \beta'_1(\eta-\eta'_0)^{-m'_1}\} & \eta \geqslant \eta'_0, \end{cases}$$

and

$$\bar{f}(\eta,\tau) = \begin{cases} e^{(1/\gamma)(s(\eta e^{\tau/2})-\bar{s})} v^+ & \eta \geqslant \eta'_1 \\ \min\{e^{(1/\gamma)(s(\eta e^{\tau/2})-\bar{s})} v^+, v^- + \beta'_2(-\eta+\eta'_1)^{-m'_2}\} & \\ = \min\{v^+, v^- + \beta'_2(-\eta+\eta'_1)^{-m'_2}\} & \eta < \eta'_1. \end{cases}$$

Similar to the proof of Lemma 2.2, in view of Lemma 2.8, it can be shown that for any positive smooth solution $v(\eta,\tau) > 0$ of (2.65), satisfying $v_\eta(\eta,\tau) \to 0$ as $|\eta| \to +\infty$, it holds that

$$\int_{-\infty}^{+\infty} [v(\eta,\tau) - \tilde{v}(\eta,\tau)]\, d\tau = e^{-\tau/2} \int_{-\infty}^{+\infty} e^{(1/\gamma)(s(x)-\bar{s})} [v^*(x+x^*) - v^*(x+x_0)]\, dx. \tag{2.68}$$

Since $v^*_\eta > 0$ for $\eta \in R$, it follows that if $x^* \geqslant x_0$, then $v^*(x+x^*) \geqslant v^*(x+x_0)$, and hence the maximum principle implies that $v(\eta, \tau) \geqslant \tilde{v}(\eta, \tau)$ for $(\eta, \tau) \in S_{T'}$, while

$$|v(\eta, \tau) - \tilde{v}(\eta, \tau)| = v(\eta, \tau) - \tilde{v}(\eta, \tau), \qquad \text{for} \quad (\eta, \tau) \in S_{T'},$$

and (2.68) implies that

$$\int_{-\infty}^{+\infty} |v(\eta, \tau) - \tilde{v}(\eta, \tau)| \, d\tau \leqslant \hat{C}_1 e^{-\tau/2}, \qquad (\eta, \tau) \in S_{T'}. \tag{2.69}$$

Similarly, if $x^* \leqslant x_0$ we have $v(\eta, \tau) \leqslant \tilde{v}(\eta, \tau)$ for $(\eta, \tau) \in S_{T'}$, while

$$|v(\eta, \tau) - \tilde{v}(\eta, \tau)| = \tilde{v}(\eta, \tau) - v(\eta, \tau) \qquad \text{for} \quad (\eta, \tau) \in S_{T'},$$

and (2.68) implies that

$$\int_{-\infty}^{+\infty} |v(\eta, \tau) - \tilde{v}(\eta, \tau)| \, d\tau \leqslant \hat{C}_2 e^{-\tau/2} \qquad (\eta, \tau) \in S_{T'}, \tag{2.70}$$

where $\hat{C}_i$ $(i = 1, 2)$ is a constant.

The above arguments imply that

$$\int_{-\infty}^{+\infty} |v(\eta, \tau) - \tilde{v}(\eta, \tau)| \, d\eta \leqslant \hat{c} e^{-\tau/2},$$

$\hat{c} > 0$ is a constant. Namely, $\int_{-\infty}^{+\infty} |v(x, t) - \tilde{v}(x, t)| \, dx \leqslant \hat{c}$, with which we are able to prove Lemma 2.7 by a similar argument as used in Lemma 2.4.

First, it has been shown that

$$\int_{-\infty}^{+\infty} |\psi(x, t)| \, dx \leqslant \hat{c}. \tag{2.71}$$

Thus,

$$\begin{aligned} \int_{-\infty}^{+\infty} \psi^2(x, t) \, dx &\leqslant \sup_{x \in R} |\psi(x, t)| \int_{-\infty}^{+\infty} |\psi(x, t)| \, dx \\ &\leqslant \sqrt{2}\, \hat{c} \left[\int_{-\infty}^{+\infty} \psi^2(x, t) \, dx \right]^{1/4} \left[\int_{-\infty}^{+\infty} \psi_x^2(x, t) \, dx \right]^{1/4}, \end{aligned}$$

which implies

$$\int_{-\infty}^{+\infty} \psi^2(x, t) \, dx \leqslant (\sqrt{2}\, \hat{c})^{4/3} \left[\int_{-\infty}^{+\infty} \psi_x^2(x, t) \, dx \right]^{1/3}. \tag{2.72}$$

We estimate $\int_{-\infty}^{+\infty} \psi_x^2(x, t)\, dx$ now. Let $\varphi(x, t) = -p(\tilde{v} + \psi, s)(x, t) + p(\tilde{v}, s)(x, t)$, $(2.59)_1$ becomes

$$\psi_t = \varphi_{xx}. \tag{2.73}$$

Define $W = \varphi_x$, it follows

$$W_t = \varphi_{xt}.$$

Multiply the above equation by W and integrate it over $[0, t] \times R$, by the same argument as used for (2.48), one obtains

$$\frac{1}{2} \int_{-\infty}^{+\infty} \varphi_x^2(x, t)\, dx + \frac{\alpha_0}{2} \int_0^t \int_{-\infty}^{+\infty} \psi_t^2(x, \tau)\, dx\, d\tau$$
$$\leqslant \frac{1}{2} \int_{-\infty}^{+\infty} \varphi_x^2(x, 0)\, dx + \frac{M}{2\alpha_0} \int_0^t \int_{-\infty}^{+\infty} \tilde{v}_t^2\, dx\, d\tau.$$

This, combined with Lemma 2.6, yields

$$\int_{-\infty}^{+\infty} \psi_x^2(x, t)\, dx + \frac{\alpha_0}{2} \int_0^t \int_{-\infty}^{+\infty} \psi_t^2(x, \tau)\, dx\, d\tau$$
$$\leqslant \hat{K}_5 \{ \hat{e}(0) + [v_\eta^*(0)]^2 + \|s(x) - \bar{s}\|_{H^5}^2$$
$$+ \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| + |v^+ - v^-| \},$$

where $\hat{K}_5$ depends only on the quantities cited in Lemma 2.4.

The proof of Lemma 2.7 can be finished then, following the same lines as in Lemma 2.4. By a similar approach as used in Lemma 2.5 and Lemma 2.6, we prove

LEMMA 2.9. *Under the assumptions of Lemma 2.5, if $|v^+ - v^-|$ and $\|s(x) - \bar{s}\|_{H^5}^2$ are suitably small, then there exists a unique globally defined smooth solution $\psi(t) \in X_3$ for Cauchy problem* (2.59). *The function $v(x, t)$, defined by $v(x, t) = \tilde{v}(x, t) + \psi(x, t)$, is the unique positive smooth solution of Cauchy problem* (2.1) *for which it holds that*

$$a_1 \leqslant e^{(1/\gamma)(s(x) - \bar{s})} v^- \leqslant v(x, t) \leqslant e^{(1/\gamma)(s(x) - \bar{s})} v^+ \leqslant a_2$$

$$\|v_x\|_{H^2}^2 + \|v_t\|_{H^2}^2 + \|v_{tt}\|_{H^1}^2 + |v_x|_{C^1}^2 + |v_t|_{C^1}^2 + |v_{tt}|_{C^0}^2$$
$$+ \int_0^t \int_{-\infty}^{+\infty} [v_t^2 + v_{xt}^2 + v_{tt}^2 + v_{xxt}^2 + v_{ttx}^2](x, \tau)\, dx\, d\tau$$
$$\leqslant G_1 \{ \|s(x) - \bar{s}\|_{H^5}^2 + \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1|$$
$$+ [v_\eta^*(0)]^2 + |v^+ - v^-| \}^{1/3} =: G_1 \hat{\varepsilon}^{1/3}$$
$$|v_{xxx}^2(\cdot, t)|_{C^0} \leqslant G_2 \hat{\varepsilon}^{1/3}$$

Furthermore, if $\psi(t) \in X_4$, *it holds also that*

$$\int_0^t \int_{-\infty}^{+\infty} [v_{xxxt}^2 + v_{xxtt}^2](x, \tau)\, dx\, d\tau \leqslant G_3 \hat{\varepsilon}^{1/3},$$

where the positive constants G_i $(i = 1, 2, 3)$ *depend only on those quantities cited in Lemma* 2.4.

3. The Main Theorem

We study the Cauchy problem (1.12)–(1.13) in this section under the assumptions introduced in Section 1 and Section 2. We assume also that $u_0(x) \in H^2(R)$ and $[v_0(x) - v^*(x)] \in H^5(R)$. By a similar approach to the one used in [HS], we solve this Cauchy problem in the space X_3 as well.

It can be proved that the classical local existence theorem gives the solution for the Cauchy problem (1.12)–(1.13) in the space X_3 locally in time. For the global existence in $t > 0$ we only need the a priori estimates in the norm $\| y(\cdot, t)\|_2$ for which the a priori estimates in the norm of X_3 is sufficient by (2.12).

Choose $\hat{r} > 0$ such that $0 < \hat{r} < a_1 (a_1 = \inf_{x \in (-\infty, \infty)} e^{(1/\gamma)(s(x) - \bar{s})} v_-)$ and introduce

$$E(t) = \sum_{j=1}^{3} E_j(t) \tag{3.1}$$

for the solution y with $y_x \in [-\hat{r}, \hat{r}]$ at each $(x, t) \in R \times [0, T]$, where

$$E_1(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \left\{ \frac{1}{2} \left(y \cdot y_t + \frac{y^2}{2} \right) + y_t^2 + 2Q(y_x, v, s) \right\} (x, t)\, dx$$

$$E_2(t) = \frac{1}{2} \int_{-\infty}^{+\infty} \{ y_{tt}^2 + [1 - p_v(y_x + v, s)]\, y_{tx}^2$$
$$+ [-p_v(y_x + v, s)]\, y_{xx}^2 \}(x, t)\, dx$$

$$E_3 = \frac{1}{2} \int_{-\infty}^{+\infty} \{ y_{ttx}^2 + y_{xxt}^2 \}(x, t)\, dx$$

and $Q = \int_0^{y_x} [p(v, s) - p(\lambda + v, s)]\, d\lambda =: q(x, t)$.

It is known from Lemma 2.9 that it holds, for $y_x \in [-\hat{r}, \hat{r}]$, that

$$0 < a_1 - \hat{r} \leqslant y_x + v \leqslant a_2 + \hat{r}.$$

It is easy to know also that there exists a positive constant $\beta_0 > 0$ such that

$$\inf_{\substack{v \in [a_1 - \hat{r},\, a_2 + \hat{r}] \\ s \in [\inf_{x \in R} s_0(x),\, \sup_{x \in R} s_0(x)]}} [-p_v(v, s)] \geqslant \beta_0 > 0.$$

We assume $\beta_0 \leqslant 1$ for convenience. Then $(\beta_0/2)\, y_x^2 \leqslant Q \leqslant (M/2)\, y_x^2$ for $y_x \in [-\hat{r}, \hat{r}]$.

It is not difficult to claim then, by Lemma 2.9, (1.12), and (2.12), that

$$\begin{aligned} \|y(t)\|_2^2 &\leqslant K \,|||y(t)|||_3^2 \\ &\leqslant \hat{D}_1 E(t) + \hat{D}_2 \{ \|s(x) - \bar{s}\|_{H^5}^2 + \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| \\ &\quad + [v_\eta^*(0)]^2 + |v^+ - v^-| \}^{1/3}, \qquad t \in [0, T], \end{aligned} \tag{3.2}$$

where $\hat{D}_1$ and $\hat{D}_2$ depend only on $\hat{r}$ and the quantities cited in Lemma 2.4.

LEMMA 3.1. *There exists a constant $\varepsilon = \varepsilon(\hat{r})$ with $0 < \varepsilon \leqslant \max\{1, \hat{r}/2\}$ such that if $y(t) \in X_3$ with $y_x \in [-\hat{r}, \hat{r}]$ is the solution of the Cauchy problem* (1.12), (1.13) *and is small as*

$$\|y(t)\|_2 < \varepsilon \qquad \text{in} \quad 0 \leqslant t \leqslant T \tag{3.3}$$

and if it holds also that

$$\|s(x) - \bar{s}\|_{H^5} < \varepsilon, \qquad |v^+ - v^-| < \varepsilon, \tag{3.4}$$

then one has the a priori estimate

$$\begin{aligned} E(t) &\leqslant F_0 E(0) + F_1 \{ \|s(x) - \bar{s}\|_{H^5}^2 + \sup_{x \in [-N, N]} |e^{(1/\gamma)(s(x) - \bar{s})} - 1| \\ &\quad + [v_\eta^*(0)]^2 + |v^+ - v^-| \}^{1/3}, \qquad \text{in} \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{3.5}$$

where F_0 and F_1 depend only on $\hat{r}$ and the quantities cited in Lemma 2.4.

For the proof, we still first assume that the solutions $y(t)$, $\theta(t)$ and $\psi(t)$ belong to the space X_4 since the a priori estimate (3.5) will be also valid for the solutions in X_3 by use of the Friedrich's mollifier under the same assumptions (3.3) and (3.4), for which we omit the detail.

Proof. Multiply the equation (1.12) by y and y_t respectively and integrate then over $[0, t] \times (-\infty, \infty)$, after the integration by parts, we arrive at

$$\int_{-\infty}^{+\infty}\left(y\cdot y_t+\frac{y^2}{2}\right)(x,t)\,dx+\int_0^t\int_{-\infty}^{+\infty}[-p_v(\sigma y_x+v,s)]\;y_x^2(x,\tau)\,dx\,d\tau$$
$$=\int_{-\infty}^{+\infty}\left(y\cdot y_t+\frac{y^2}{2}\right)(x,0)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_t^2(x,\tau)\,dx\,d\tau$$
$$+\int_0^t\int_{-\infty}^{+\infty}[-p_v(v,s)\,v_t\,y_x](x,\tau)\,dx\,d\tau \qquad 0<\sigma<1. \tag{3.6}$$

$$\int_{-\infty}^{+\infty}\left\{\frac{y_t^2}{2}+q\right\}(x,t)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,d\tau$$
$$=\int_{-\infty}^{+\infty}\left\{\frac{y_t^2}{2}+q\right\}(x,0)\,dx$$
$$+\int_0^t\int_{-\infty}^{+\infty}y_t[p_{vv}(v,s)\,v_xv_t+p_{vs}(v,s)\,v_t\cdot s'+p_v(v,s)\,v_{tx}]\,dx\,d\tau$$
$$-\int_0^t\int_{-\infty}^{+\infty}\{[p_v(v,s)\,y_x-p(v+y_x,s)+p(v,s)]\,v_t\}(x,\tau)$$
$$\times\frac{y_x^2}{2}[p_v(\sigma y_x+v,s)]_t\,(x,\tau)\,dx\,d\tau. \tag{3.7}$$

By using Cauchy inequality with (3.6) and (3.7), it follows that

$$\int_{-\infty}^{+\infty}\left(y\cdot y_t+\frac{y^2}{2}\right)(x,t)\,dx+\beta_0\int_0^t\int_{-\infty}^{+\infty}y_x^2(x,\tau)\,dx\,d\tau$$
$$\leqslant\int_{-\infty}^{+\infty}\left(y\cdot y_t+\frac{y^2}{2}\right)(x,0)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_t^2(x,\tau)\,dx\,d\tau$$
$$+\frac{\beta_0}{4}\int_0^t\int_{-\infty}^{+\infty}y_x^2\,dx\,d\tau+\frac{M^2}{\beta_0}\int_0^t\int_{-\infty}^{+\infty}v_t^2\,dx\,d\tau \tag{3.8}$$

and

$$\int_{-\infty}^{+\infty}\left\{\frac{y_t^2}{2}+Q\right\}(x,t)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,d\tau$$
$$\leqslant\int_{-\infty}^{+\infty}\left\{\frac{y_t^2}{2}+Q\right\}(x,0)\,dx+\frac{1}{4}\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,d\tau$$
$$+M^2\int_0^t\int_{-\infty}^{+\infty}v_{xt}^2\,dx\,d\tau+\delta_1\int_0^t\int_{-\infty}^{+\infty}\{y_t^2+y_x^2+v_t^2\}\,dx\,d\tau. \tag{3.9}$$

Hereafter δ_1 is a small quantity satisfying

$$\begin{aligned}\delta_1 &\leqslant A_1[\,\|y(t)\|_2+\|v(t)\|_2+\|s(x)-\bar{s}\|_{H^5}]\\ &\leqslant A_2\{\,\|y(t)\|_2+[\,\|s(x)-\bar{s}\|^2_{H^5}+\sup_{x\in[-N,\,N]}|e^{(1/\gamma)(s(x)-\bar{s})}-1|\\ &\quad+[v^*_\eta(0)]^2+|v^+-v^-|\,]^{1/6}\},\end{aligned}\tag{3.10}$$

where $A_i>0$ $(i=1,2)$ depends only on $\hat{r}$ and the quantities cited in Lemma 2.4. Therefore, it reads that

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left\{\frac{1}{4}\left[y\cdot y_t+\frac{y^2}{2}\right]+\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\,y_x^2\right\}(x,t)\,dx\\ &\quad+\left(\frac{3\beta_0}{16}-\delta_1\right)\int_0^t\int_{-\infty}^{+\infty}y_x^2\,dx\,d\tau+\left(\frac{1}{2}-\delta_1\right)\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,d\tau\\ &\leqslant\int_{-\infty}^{+\infty}\left\{\frac{1}{4}\left[y\cdot y_t+\frac{y^2}{2}\right]+\frac{y_t^2}{2}+\frac{[-p_v(\sigma y_x+v,s)]}{2}\cdot y_x^2\right\}(x,0)\,dx\\ &\quad+M^2\int_0^t\int_{-\infty}^{+\infty}v_{tx}^2\,dx\,d\tau+\left(\frac{M^2}{4\beta_0}+\delta_1\right)\int_0^t\int_{-\infty}^{+\infty}v_t^2\,dx\,d\tau.\end{aligned}\tag{3.11}$$

It is clear that there exists an $\varepsilon>0$ such that if (3.3) and (3.4) are true, then

$$\delta_1\leqslant\min\left\{\frac{\beta_0}{16},\frac{1}{4}\right\}=\frac{\beta_0}{16}.\tag{3.12}$$

Thus, it follows from (3.11), (3.12) that

$$\begin{aligned}&E_1(t)+\frac{\beta_0}{8}\int_0^t\int_{-\infty}^{+\infty}y_x^2\,dx\,d\tau+\frac{1}{4}\int_0^t\int_{-\infty}^{+\infty}y_t^2\,dx\,d\tau\\ &\quad\leqslant E_1(0)+M^2\int_0^t\int_{-\infty}^{+\infty}v_{tx}^2\,dx\,d\tau+\left(\frac{M^2}{4\beta_0}+\delta_1\right)\int_0^t\int_{-\infty}^{+\infty}v_t^2\,dx\,d\tau.\end{aligned}\tag{3.13}$$

Differentiate (1.12) with respect to t and multiply by y_{tt} then, we obtain the following equation by integration

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\,y_{tx}^2\right\}(x,t)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_{tt}^2\,dx\,d\tau\\ &\quad=\int_{-\infty}^{+\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\,y_{tx}^2\right\}(x,0)\,dx\end{aligned}$$

$$-\int_0^t\int_{-\infty}^{+\infty}[p_v(y_x+v,s)]_t\cdot\frac{y_{tx}^2}{2}\,dx\,d\tau$$

$$-\int_0^t\int_{-\infty}^{+\infty}y_{tt}\{[p_v(y_x+v,s)-p_v(v,s)]v_t\}_x\,dx\,d\tau$$

$$+\int_0^t\int_{-\infty}^{+\infty}y_{tt}\{p_v(v,s)\,v_t\}_{xt}\,dx\,d\tau. \tag{3.14}$$

Differentiate (1.12) with respect to x and multiply by y_{xt} then, we obtain the equation below by integration

$$\int_{-\infty}^{+\infty}\left\{\frac{y_{tx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\,y_{xx}^2\right\}(x,t)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_{tx}^2\,dx\,d\tau$$

$$=\int_{-\infty}^{+\infty}\left\{\frac{y_{tx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\,y_{xx}^2\right\}(x,0)\,dx$$

$$-\int_0^t\int_{-\infty}^{+\infty}[p_v(y_x+v,s)]_t\cdot\frac{y_{xx}^2}{2}\,dx\,d\tau$$

$$-\int_0^t\int_{-\infty}^{+\infty}y_{tx}\{[p_v(y_x+v,s)-p_v(v,s)]v_x$$

$$+[p_s(y_x+v,s)-p_s(v,s)]s'\}_x\,dx\,d\tau$$

$$+\int_0^t\int_{-\infty}^{+\infty}y_{tx}\{p_v(v,s)v_t\}_{xx}\,dx\,d\tau. \tag{3.15}$$

By using Cauchy inequality to (3.14) and (3.15), it yields that

$$\int_{-\infty}^{+\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[1-p_v(y_x+v,s)]}{2}\,y_{tx}^2+\frac{[-p_v(y_x+v,s)]}{2}\,y_{xx}^2\right\}(x,t)\,dx$$

$$+\left(\frac{1}{2}-\delta_1\right)\int_0^t\int_{-\infty}^{+\infty}(y_{tt}^2+y_{tx}^2)\,dx\,d\tau$$

$$\leqslant\int_{-\infty}^{+\infty}\left\{\frac{y_{tt}^2}{2}+\frac{[1-p_v(y_x+v,s)]}{2}\,y_{tx}^2+\frac{[-p_v(y_x+v,s)]}{2}\,y_{xx}^2\right\}(x,0)\,dx$$

$$+\frac{M^2}{2}\int_0^t\int_{-\infty}^{+\infty}(v_{ttx}^2+v_{xxt}^2)\,dx\,d\tau$$

$$+\delta_1\int_0^t\int_{-\infty}^{+\infty}(y_t^2+y_x^2+v_t^2+v_{tx}^2)\,dx\,d\tau. \tag{3.16}$$

Then, (3.13) and (3.16) imply

$$
\begin{aligned}
E_2(t)&+\frac{1}{4}\int_0^t\int_{-\infty}^{+\infty}(y_{tt}^2+y_{tx}^2)\,dx\,d\tau\\
&\leqslant E_2(0)+\frac{M^2}{2}\int_0^t\int_{-\infty}^{+\infty}(v_{ttx}^2+v_{xxt}^2)\,dx\,d\tau\\
&\quad+\delta_1\int_0^t\int_{-\infty}^{+\infty}(v_t^2+v_{tx}^2)\,dx\,d\tau\\
&\quad+\frac{\delta_1}{(\beta_0/8)+(1/4)}\left\{E_1(0)+M^2\int_0^t\int_{-\infty}^{+\infty}v_{tx}^2\,dx\,d\tau\right.\\
&\quad\left.+\left(\frac{M^2}{4\beta_0}+\delta_1\right)\int_0^t\int_{-\infty}^{+\infty}v_t^2\,dx\,d\tau\right\}.
\end{aligned}
$$

Namely,

$$
\begin{aligned}
E_2(t)&+\tfrac{1}{4}\int_0^t\int_{-\infty}^{+\infty}(y_{tt}^2+y_{tx}^2)\,dx\,d\tau\\
&\leqslant E_2(0)+F_2E_1(0)+F_3\int_0^t\int_{-\infty}^{+\infty}[v_{ttx}^2+v_{xxt}^2+v_{tx}^2+v_t^2]\,dx\,d\tau, \qquad (3.17)
\end{aligned}
$$

where the constants F_2 and F_3 depend on $\hat{r}$ and the quantities cited in Lemma 2.4.

Differentiate (1.12) with respect to x and t successively and multiply the resulting equation by y_{ttx}, integrate it then over $[0,t]\times(-\infty,\infty)$, one obtains the following inequality by using (1.12) and differentiating (1.12) with respect to x.

$$
\begin{aligned}
\int_{-\infty}^{+\infty}&\left\{y_{ttx}^2+\frac{[-p_v(y_x+v,s)]}{2}\,y_{xxt}^2\right\}(x,t)\,dx+\int_0^t\int_{-\infty}^{+\infty}y_{ttx}^2\,dx\,d\tau\\
&\leqslant\int_{-\infty}^{+\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{[-p_v(y_x+v,s)]}{2}\,y_{xxt}^2\right\}(x,0)\,dx\\
&\quad+\int_0^t\int_{-\infty}^{+\infty}|(-p_v(v,s))\,y_{ttx}\cdot v_{ttxx}|\,dx\,d\tau\\
&\quad+\delta_1\int_0^t\int_{-\infty}^{+\infty}(y_{ttx}^2+y_{xxt}^2+y_{tt}^2+y_{tx}^2+y_t^2+y_x^2)\,dx\,d\tau\\
&\quad+\delta_1\int_0^t\int_{-\infty}^{+\infty}(v_{ttx}^2+v_{txx}^2+v_{tt}^2+v_{tx}^2+v_t^2)\,dx\,d\tau. \qquad (3.18)
\end{aligned}
$$

To estimate the term of $\int_0^t \int_{-\infty}^{+\infty} y_{xxt}^2$, we differentiate (1.12) with respect to t and multiply it by y_{xxt}, integrate over $[0, t] \times (-\infty, \infty)$ then, we arrive at

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} [-p_v(y_x+v, s)] \cdot y_{xxt}^2 \, dx \, d\tau \\
&\quad = \int_{-\infty}^{+\infty} (y_{tt} \cdot y_{xxt})(x, t) \, dx \\
&\qquad - \int_{-\infty}^{+\infty} (y_{tt} \cdot y_{xxt})(x, 0) \, dx + \int_0^t \int_{-\infty}^{+\infty} y_{ttx}^2 \, dx \, d\tau \\
&\qquad + \int_0^t \int_{-\infty}^{+\infty} y_{xxt} \{ y_{tt} + [p_v(y_x+v, s)]_t \cdot y_{xx} \\
&\qquad + [p_v(y_x+v, s) - p_v(v, s)]_t \cdot v_x \\
&\qquad + [p_v(y_x+v, s) - p_v(v, s)] \, v_{xt} + [p_s(y_x+v, s) - p_s(v, s)]_t \cdot s' \\
&\qquad - [p_v(v, s) v_x + p_s(v, s) s']_{tt} \} \, dx \, d\tau \\
&\quad \leqslant \tfrac{1}{2} \int_{-\infty}^{+\infty} y_{tt}^2(x, t) \, dx + \tfrac{1}{2} \int_{-\infty}^{+\infty} y_{xxt}^2(x, t) \, dx - \int_{-\infty}^{+\infty} (y_{tt} \cdot y_{xxt})(x, 0) \, dx \\
&\qquad + \int_0^t \int_{-\infty}^{+\infty} y_{ttx}^2 \, dx \, d\tau + \int_0^t \int_{-\infty}^{+\infty} | y_{xxt} \{ y_{tt} - p_v(v, s) \, v_{xtt} \} | \, dx \, d\tau \\
&\qquad + \delta_1 \int_0^t \int_{-\infty}^{+\infty} [y_{xxt}^2 + y_{xt}^2 + y_x^2 + v_{tt}^2 + v_t^2] \, dx \, d\tau.
\end{aligned}
$$

This, together with (3.12) and (3.18), implies

$$
\begin{aligned}
&\frac{1}{4} \int_0^t \int_{-\infty}^{+\infty} y_{xxt}^2 \, dx \, d\tau \\
&\qquad \leqslant F_4 [E_1(0) + E_2(0) + E_3(0)] \\
&\qquad\quad + \frac{1}{2\beta_0} \int_{-\infty}^{+\infty} y_{xxt}^2(x, t) \, dx + \frac{1}{\beta_0} \int_0^t \int_{-\infty}^{+\infty} y_{ttx}^2 \, dx \, d\tau \\
&\qquad\quad + F_5 \int_0^t \int_{-\infty}^{+\infty} (v_{ttx}^2 + v_{xxt}^2 + v_{tx}^2 + v_{tt}^2 + v_t^2) \, dx \, d\tau. \qquad (3.19)
\end{aligned}
$$

Substituting (3.19) into (3.18), it reads, with the help of (3.17), that

$$\int_{-\infty}^{+\infty}\left\{\frac{y_{ttx}^2}{2}+\left(\frac{\beta_0}{2}-2\beta_0\delta_1\right)y_{xxt}^2\right\}(x,t)\,dx$$
$$+\left(\frac{1}{2}-\delta_1-\frac{4\delta_1}{\beta_0}\right)\int_0^t\int_{-\infty}^{+\infty}y_{ttx}^2\,dx\,d\tau$$
$$\leqslant F_6E(0)+F_7\int_0^t\int_{-\infty}^{+\infty}(v_{ttx}^2+v_{txx}^2+v_{tt}^2+v_{tx}^2+v_t^2+v_{ttxx}^2)\,dx\,d\tau. \tag{3.20}$$

It is obvious that there exists an $\varepsilon(\hat{r})$ with $0<\varepsilon(\hat{r})\leqslant 1$ such that if (3.3) and (3.4) are true, then

$$\delta_1\leqslant\frac{\beta_0^2}{16}. \tag{3.21}$$

Thus, (3.20) implies, since $\beta_0\leqslant 1$, that

$$\int_{-\infty}^{+\infty}\left\{\frac{y_{ttx}^2}{2}+\frac{3\beta_0}{8}\,y_{xxt}^2\right\}(x,t)\,dx+\frac{3\beta_0}{16}\int_0^t\int_{-\infty}^{+\infty}y_{ttx}^2\,dx\,d\tau$$
$$\leqslant F_6E(0)+F_7\int_0^t\int_{-\infty}^{+\infty}(v_{ttx}^2+v_{txx}^2+v_{tt}^2+v_{tx}^2+v_t^2+v_{ttxx}^2)\,dx\,d\tau, \tag{3.22}$$

where the constant F_i $(i=4, ..., 7)$ depends only on $\hat{r}$ and the quantities cited in Lemma 2.4.

Lemma 3.1 follows then from (2.14), (3.17), (3.22), and Lemma 2.9.

THEOREM 3.2. *Under the hypotheses* (1.7)–(1.9) *and Condition* V, *the Cauchy problem* (1.12), (1.13) *has a unique smooth solution in the large in time, provided that* $E(0)$, $|v^+-v^-|$ *and* $\|s(x)-\bar{s}\|_{H^5}$ *are suitably small. Moreover, the solution* $y(t)$ *decays to zero in the* L_∞*-norm as* $t\to+\infty$ *and so do its first derivatives.*

Proof. We choose the initial data so small that

$$2\hat{D}_1E(0)+\hat{D}_2[\,\|s_0(x)-\bar{s}\|_{H^5}^2+\sup_{x\in[-N,N]}|e^{(1/\gamma)(s(x)-\bar{s})}-1|$$
$$+[v_\eta^*(0)]^2+|v^+-v^-|\,]^{1/3}<\varepsilon^2 \tag{3.23}$$
$$2\hat{D}_1F_1E(0)+(2\hat{D}_1F_2+\hat{D}_2)[\,\|s_0(x)-\bar{s}\|_{H^5}^2+\sup_{x\in[-N,N]}|e^{(1/\gamma)(s(x)-\bar{s})}-1|$$
$$+[v_\eta^*(0)]^2+|v^+-v^-|\,]^{1/3}<\varepsilon^2 \tag{3.24}$$

and the requirements cited in Lemma 2.5 and Lemma 2.7 are satisfied respectively, where $\hat{D}_1$ and $\hat{D}_2$ denote the same constants as in (3.2), while ε and F_i $(i=1,2)$ are the same as in Lemma 3.1.

At first, in view of (3.2) we get

$$|y_x(x,0)|^2 \leqslant \|y(x,0)\|_2^2 \leqslant \hat{D}_1 E(0) + \hat{D}_2\{\|s(x)-\bar{s}\|_{H^5}^2 + \sup_{[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})}-1| + [v_\eta^*(0)]^2 + |v^+ - v^-|\}^{1/3} < \varepsilon^2 \leqslant \frac{\hat{r}^2}{4},$$

then $y_x(x,0) \in [-\hat{r}/2, \hat{r}/2]$, $x \in R$. By the local existence theorem, there exists a positive t_1 such that the solution $y(t)$ exists in $0 \leqslant t \leqslant t_1$ and satisfies

$$E(t) \leqslant 2E(0) \quad \text{and} \quad y_x(x,t) \in [-\hat{r}, \hat{r}] \quad \text{for} \quad (x,t) \in R \times [0, t_1]. \tag{3.25}$$

It follows from (3.2) then that

$$\|y(t)\|_2^2 \leqslant 2\hat{D}_1 E(0) + \hat{D}_2[\|s_0(x)-\bar{s}\|_{H^5}^2 + \sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})}-1| + [v_\eta^*(0)]^2 + |v^+ - v^-|]^{1/3} < \varepsilon^2 \quad \text{in} \quad 0 \leqslant t \leqslant t_1 \tag{3.26}$$

and

$$y_x(x,t) \in \left[-\frac{\hat{r}}{2}, \frac{\hat{r}}{2}\right] \quad \text{in} \quad 0 \leqslant t \leqslant t_1 \tag{3.27}$$

since $\varepsilon \leqslant \max\{1, \hat{r}/2\}$.

Thus, Lemma 3.1 implies

$$E(t) \leqslant F_1 E(0) + F_2[\|s_0(x)-\bar{s}\|_{H^5}^2 + \sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})}-1| + [v_\eta^*(0)]^2 + |v^+ - v^-|]^{1/3} \quad \text{in} \quad 0 \leqslant t \leqslant t_1. \tag{3.28}$$

Next, by the local existence theorem for $t \geqslant t_1$, there exists a positive $\bar{t}$ such that the solution $y(t)$ exists in $0 \leqslant t \leqslant t_1 + \tilde{t}$ and satisfies

$$E(t) \leqslant 2E(t_1), \quad y_x(x,t) \in [-\hat{r}, \hat{r}], \quad \text{in} \quad t_1 \leqslant t \leqslant t_1 + \tilde{t}. \tag{3.29}$$

In view of (3.2), (3.24), (3.28), and (3.29), it reads that

$$\begin{aligned}\|y(t)\|_2^2 &\leqslant \hat{D}_1 E(t) + \hat{D}_2 [\|s(x)-\bar{s}\|_{H^5}^2 + \sup_{[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| \\ &\quad + [v_\eta^*(0)]^2 + |v^+ - v^-|]^{1/3} \\ &< \varepsilon^2 \qquad \text{in} \quad t_1 \leqslant t \leqslant t_1 + \tilde{t}\end{aligned} \tag{3.30}$$

and therefore

$$y_x(x,t) \in \left[-\frac{\hat{r}}{2}, \frac{\hat{r}}{2}\right] \qquad \text{in} \quad t_1 \leqslant t \leqslant t_1 + \tilde{t}. \tag{3.31}$$

Thus, Lemma 3.1 implies

$$\begin{aligned}E(t) &\leqslant F_1 E(0) + F_2 [\|s(x)-\bar{s}\|_{H^5}^2 + \sup_{x\in[-N,N]} |e^{(1/\gamma)(s(x)-\bar{s})} - 1| \\ &\quad + [v_\eta^*(0)]^2 + |v^+ - v^-|]^{1/3} \qquad \text{in} \quad 0 \leqslant t \leqslant t_1 + \tilde{t}.\end{aligned} \tag{3.32}$$

Repeat the same procedure with the same time interval $\tilde{t} > 0$, we complete the proof of the global existence of the solution.

We prove the decay of the solution now.

By the above argument and Lemma 3.1, it follows that

$$\int_{-\infty}^{+\infty} (y_t^2 + y_x^2)(x,t)\, dx \leqslant \Lambda$$

and

$$\begin{aligned}&\int_0^{+\infty} \left| \frac{d}{dt} \int_{-\infty}^{+\infty} y_t^2(x,t)\, dx \right| dt \\ &\quad \leqslant \int_0^{+\infty} \int_{-\infty}^{+\infty} y_t^2(x,t)\, dx\, dt + \int_0^{+\infty} \int_{-\infty}^{+\infty} y_{tt}^2(x,t)\, dx\, dt \leqslant \Lambda,\end{aligned}$$

where $\Lambda > 0$ is a constant, independent of t.

These imply that

$$\int_{-\infty}^{+\infty} y_t^2(x,t)\, dx \to 0 \qquad \text{as} \quad t \to +\infty.$$

Similarly, it can be shown that $\int_{-\infty}^{+\infty} y_x^2(x,t)\, dx \to 0$ as $t \to +\infty$. On the other hand, it is known that

$$y^2(x,t) \leqslant 2 \left[\int_{-\infty}^{+\infty} y^2(x,t)\, dx \right]^{1/2} \left[\int_{-\infty}^{+\infty} y_x^2(x,t)\, dx \right]^{1/2} \leqslant \Lambda \left[\int_{-\infty}^{+\infty} y_x^2\, dx \right]^{1/2}.$$

Therefore, $y(x,t) \to 0$ as $t \to +\infty$ uniformly for $x \in (-\infty, \infty)$.

Next, it reads from Lemma 3.1 that

$$\int_{-\infty}^{+\infty} (y_{tt}^2 + y_{tx}^2)(x, t)\, dx \leqslant \Lambda$$

and

$$\int_0^\infty \left| \frac{d}{dt} \int_{-\infty}^{+\infty} y_{tx}^2(x, t)\, dx \right| dt$$

$$\leqslant \int_0^{+\infty} \int_{-\infty}^{+\infty} y_{tx}^2(x, t)\, dx\, dt + \int_0^{+\infty} \int_{-\infty}^{+\infty} y_{ttx}^2(x, t)\, dx\, dt \leqslant \Lambda.$$

These imply

$$\int_{-\infty}^{+\infty} y_{tx}^2(x, t)\, dx \to 0 \qquad \text{as} \quad t \to +\infty.$$

Since

$$y_t^2(x, t) \leqslant 2 \left[\int_{-\infty}^{+\infty} y_t^2(x, t)\, dx \right]^{1/2} \left[\int_{-\infty}^{+\infty} y_{tx}^2(x, t)\, dx \right]^{1/2}$$

$$\leqslant \Lambda \left[\int_{-\infty}^{+\infty} y_{tx}^2\, dx \right]^{1/2},$$

it becomes

$$y_t(x, t) \to 0 \qquad \text{as} \quad t \to +\infty \qquad \text{uniformly for} \quad x \in (-\infty, \infty).$$

A similar argument shows $y_x(x, t) \to 0$ as $t \to \infty$, uniformly to $x \in (-\infty, \infty)$.

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